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ELEMENTARY ALGEBRA AND MENSURATION

INSTRUCTION PAPER

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ELEMENTARY ALGEBRA.

1. Algebra is a branch of mathematics in which **quantities** are represented by **letters** and *processes* by *signs*. It may be considered an extension of Arithmetic.

SYMBOLS.

The quantities and processes used in algebra are represented by symbols.

2. **Symbols of Quantity** are the letters of the alphabet and the figures used in Arithmetic.

The first letters of the alphabet and figures are used to represent those quantities whose values are *known*.

Unknown Quantities are those whose values are to be found and are represented by the last letters of the alphabet.

3. **Symbols of Operation** indicate that some process is to be performed.

4. The sign of *addition* $+$ is called **plus**; thus, $3+4$ is read "3 plus 4" and means that 4 is to be added to 3. Similarly, $a+b$ means that the quantity represented by b is to be added to the quantity represented by a.

5. The sign of *subtraction* $-$ is called **minus**; thus, $e-f$ is read "e minus f" and means that f is to be subtracted from e.

6. *Multiplication* is expressed by the sign $\times$ which is read "multiplied by" or "times."

7. *Division* may be expressed by the sign $\div$ or by writing the dividend above and the divisor below a horizontal line. Thus,

$$a \times c \div (e+f) \text{ is the same as } \frac{a \times c}{e+f}.$$

8. These signs are the same as those used in Arithmetic. In multiplication, however, there is a difference. In Arithmetic, the sign × must *always* be used, but in Algebra it is frequently omitted.

Thus,

$$\frac{a\,b\,m\,y}{x\,y} \text{ is the same as } \frac{a \times b \times m \times y}{x \times y}.$$

If, however, we omit the sign between the figures in the expression, $\frac{2 \times 3 \times 16}{2 \times 3}$, the value is entirely different.

$$\frac{2 \times 3 \times 16}{2 \times 3} = 16 \qquad \frac{2316}{23} = 100.7 \text{ (nearly).}$$

We can write $6 \times a \times b$ as $6\,a\,b$. Also, $12 \times 4 \times m \times n$ as $12 \times 4\,m\,n$ but not $124\,m\,n$.

9. We may say: *The sign of multiplication may be omitted between letters and between letters and figures but* **not between figures.** When there is no sign between letters multiplication is understood. For this reason the other signs *must* be used.

The only exception to the above statement is found in the case of the + sign; it may be omitted at the beginning. Thus, $2\,a + 3\,b$ is the same as $+ 2\,a + 3\,b$. The sign of $2\,a$ is understood to be +.

10. Sometimes a point or period is used in place of the multiplication sign; thus,

$$a.\,b.\,c.\,d. = a \times b \times c \times d = a\,b\,c\,d.$$

11. A *Coefficient* is a number prefixed or placed before a quantity to indicate how many times the quantity is to be taken. Thus, in $3\,a\,b$, 3 is the coefficient and shows that $a\,b$ is to be taken 3 times. In other words $3\,a\,b$ is the same as $a\,b + a\,b + a\,b$.

Let us consider the expression $6\,m\,n$. The coefficient of $m\,n$ is 6 and $6\,m$ is considered the coefficient of n. Similarly, $6\,n$ is the coefficient of m.

When no coefficient is expressed, it is understood to be 1. Thus c is the same as $1\,c$ because c multiplied by 1 is c.

12. An *Exponent* is a figure or letter written at the right and a little above a quantity to indicate the number of times the quantity is to be taken as a *factor*. For example, in c^4 the ^{4} shows that c is to be taken 4 times as a factor. c^4 is equivalent to $c \times c \times c \times c$.

13. When no exponent is written, the ***first power*** is understood; b is the same as b^1.

b^2 is "b to the second power" or b square; it is equal to $b\,b$.
b^3 is "b to the third power" or b cube; it is equal to $b\,b\,b$.
b^4 is "b to the fourth power" and is equal to $b\,b\,b\,b$.

The distinction between coefficient and exponent must be thoroughly understood. In the case of the coefficient, the coefficient shows how many times the quantity is taken to make a given *sum*. The exponent indicates how many times the quantity is to be taken to make a given *product*. Thus,

$$4\,a = a + a + a + a.$$
$$a^4 = a \times a \times a \times a.$$

14. Symbols of Relation are the signs which show equality and inequality.

The sign $=$ is read "equals." The expression $a = b$ means that the quantity represented by a is equal to the quantity represented by b.

When two quantities are connected by the $=$ sign, the expression is called an *equation*.

The sign $>$ is read "is greater than" and the sign $<$ "is less than." Thus, $a > b$ means that the quantity represented by a is greater than the quantity represented by b. $m < n$ means that m is less than n.

The sign $=\!\!\circ\!\!=$ means "is equivalent to."

15. Symbols of Abbreviation. The following are the most common symbols of abbreviation :

Radical Sign	$\sqrt{\quad}$	**Braces**	$\{\quad\}$
Parentheses	$(\quad)$		
Brackets	$[\quad]$	**Vinculum**	$\overline{\qquad\qquad}$

16. We learned the meaning of the $\sqrt{\quad}$ when studying Arithmetic.

In Algebra the parentheses, brackets, braces and vinculum are used in the same way as in Arithmetic and have exactly the same meaning.

If several quantities are enclosed in parentheses, brackets, etc., they are to be taken as a *single quantity*. For example,

$$7 - 3 + (8 - 2).$$

This expression means that 3 is to be subtracted from 7 and the result thus obtained added to the difference between 8 and 2.

$$7 - 3 + (8 - 2) = ?$$
$$7 - 3 + 6 = 10$$

Suppose we have $12 - 6 - (3 - 1) + 22 - (12 + 1)$.

We may first simplify the quantities in parentheses. $(3 - 1) = 2$, and $(12 + 1) = 13$. Then we may write:

$$12 - 6 - (3 - 1) + 22 - (12 + 1) =$$
$$12 - 6 - 2 + 22 - 13 = 13$$

17. The same rules apply to letters. For example, $4 + a + (b - c)$ means that the difference between b and c is to be added to the sum of 4 and a.

We know that when there is no sign between letters they are to be multiplied. Thus $6\,a\,(x + y)$ means that $x + y$ is to be multiplied by $6\,a$, the result being $6\,a\,x + 6\,a\,y$. In other words both x and y are to be multiplied by the $6\,a$.

18. Suppose we wish to add 7 and $10\,a$ and subtract $3\,b$; we can write it in any of the following ways:

$$(7 + 10\,a - 3\,b)$$
$$[7 + 10\,a - 3\,b]$$
$$\{7 + 10\,a - 3\,b\}$$
$$\overline{7 + 10\,a - 3\,b}.$$

The expression $4 \times \overline{3 + 4 + 2}$ is equivalent to $4 \times 9 = 36$. The vinculum is most used with the root signs, thus $\sqrt[3]{9 \times 3\,a \times b}$ means the cube root of $27\,a\,b$. The expression $\sqrt{a + b}$ indicates the square root of the expression $a + b$, not $\sqrt{a} + \sqrt{b}$. Let us note the difference; let $a = 9$ and $b = 16$. Then,

$$\sqrt{a + b} = \sqrt{9 + 16} = \sqrt{25} = 5$$
$$\sqrt{a} + \sqrt{b} = \sqrt{9} + \sqrt{16} = 3 + 4 = 7$$

EXAMPLES FOR PRACTICE.

SIMPLIFY,

1.	$(14 + 13) \times (5 - 2) = ?$	81 Ans.
2.	$25 - (6 + 7) = ?$	12 Ans.
3.	$(15 - 5) \div 2a = ?$	$\frac{5}{a}$ Ans.
4.	$(18 - 3 + 6 + 7) \div (11 - 4) = ?$	4 Ans.
5.	$(12 + \overline{(7 - 3) \div 2}) \times 6m = ?$	$84m$ Ans.

ALGEBRAIC EXPRESSIONS.

An algebraic expression is a combination of letters and signs; such as $2a + ax - b$.

19. A *term* is an algebraic expression none of whose parts are separated by the + or — signs. In the above expression, $2a$, ax and b are terms.

Positive terms are those preceded by a + sign. If no sign is expressed the plus sign is understood and the term is positive. Thus in the above expression $2a$ and ax are positive terms. $6lm$ is a positive term.

Negative terms are those preceded by a — sign. Thus in the above expression $-b$ is a negative term. *This sign must never be omitted.* Equal terms with unlike signs cancel each other; thus

$$2c - 2c = 0.$$

20. A **monomial** is an expression containing but *one* term, as $7d$. If there are two terms the expression is called a **binomial.** Expressions of more than one term are called **polynomials.** Thus $x + 3b - y$ is a polynomial.

21. **Similar terms,** sometimes called like terms, are those differing only in numerical coefficient. Thus $5ab$, $8ab$ and $3ab$ are similar terms. $4ab$ and $7b$ are *unlike terms.*

NEGATIVE QUANTITIES.

22. The signs + and — are used to indicate addition and subtraction. They are also used to distinguish between quantities opposite in character. Plus quantities are those greater than zero and minus quantities those less than zero. A thermometer shows + and — temperatures or temperatures above and below zero.

A man's property may be called + and his debts — . If a man owes $100 we say that his property is — $100; if he has $75 his property may be expressed as + $75 or simply $75.

If a thermometer inside a house indicates 60° and one outside indicates 10° below zero, the difference is 70° because the temperature outside must first rise 10° to become zero and then 60° more, making a total of 70°.

By means of the same reasoning, we see that a man that has $50 is $100 better off than the man who owes $50.

ADDITION.

23. Addition is the process of finding the aggregate or sum of two or more quantities. Thus the sum of 4 and 6 is 10. The sum of $5\,a$ and $6\,a$ is $11\,a$.

If we add 5 bolts, 8 bolts, 9 bolts and 3 bolts, the sum is 25 bolts. If we let b represent one bolt we add $5\,b$, $8\,b$, $9\,b$ and $3\,b$. The result is $25\,b$.

In the same way $-3\,a$ $-6\,a$ $-9\,a$ and $-13\,a$ are equal to $-31\,a$.

24. When the terms are similar and the signs alike, *add the coefficients and to the sum annex the letter or letters; the sign of the sum will be the common sign.*

25. If the terms do not have the same signs but are similar *add together all the + quantities and all the — quantities and find the difference. The sign of the result will be the sign of the greater sum.*

$13\,a\,c^2 + 6\,a\,c^2 - a\,c^2 + 4\,a\,c^2 - 8\,a\,c^2 = 23\,a\,c^2 - 9\,a\,c^2 = 14\,a\,c^2$.

(1)	(2)	(3)	(4)	(5)
$5\,a\,x$	$-\ 3\,x^2$	$+\ 12\,x\,y$	$-\ 4\,c$	$-\ 6\,(x+y)$
$3\,a\,x$	$-\ 7\,x^2$	$-\ 17\,x\,y$	$+\ 3\,c$	$9\,(x+y)$
$8\,a\,x$	$-\ x^2$	$+\ 13\,x\,y$	$-\ 9\,c$	$-\ 2\,(x+y)$
$2\,a\,x$	$-\ 4\,x^2$	$+\ 7\,x\,y$	$-\ 6\,c$	$(x+y)$
$a\,x$	$-\ 6\,x^2$	$-\ 6\,x\,y$	$+\ 4\,c$	$-\ 13\,(x+y)$
$19\,a\,x$	$-\ 21\,x^2$	$+\ 9\,x\,y$	$-\ 12\,c$	$-\ 11\,(x+y)$

26. The sum of a and $2\,a$ is $3\,a$, but the sum of $3\,a$ and $7\,c$

is neither $10\,a$ nor $10\,c$ but is $3\,a + 7\,c$. Hence if the terms are unlike they cannot be added; *the addition is indicated by connecting them with the proper sign.*

27. Suppose we wish to add several terms. For instance, $8\,am + 7\,cm - 2\sqrt{x} - x - 3\,am + 4\,x + 5\sqrt{x} - 3\,am + cm - 3\,x - \sqrt{x} - a\,m$. We can write similar terms under each other and add them using the principles already learned.

$$\begin{array}{rrrr}
8\,am & +\,7\,cm & -\,2\sqrt{x} & -\;\;x \\
-\,3\,am & +\;\;cm & +\,5\sqrt{x} & +\,4\,x \\
-\,3\,am & & -\;\;\sqrt{x} & -\,3\,x \\
-\;\;am & & & \\
\hline
am & +\,8\,cm & +\,2\sqrt{x} &
\end{array}$$

$$\begin{array}{ccc}
(1) & (2) & (3) \\
\begin{array}{r}
7\,x - 3\,y \\
-3\,x + \;\;y \\
2\,x + 6\,y \\
\hline
6\,x + 4\,y
\end{array}
&
\begin{array}{r}
5\,a + 3\,c - \;\;4 \\
-7\,a - \;\;c + 12 \\
4\,a - 6\,c \qquad \\
\hline
2\,a - 4\,c + \;\;8
\end{array}
&
\begin{array}{r}
3\,a\,b + 5\,c \\
8\,a\,b + 2\,c \\
-3\,a\,b \qquad \\
\hline
8\,a\,b + 7\,c
\end{array}
\end{array}$$

EXAMPLES FOR PRACTICE.

1. Add together $3\,a\,x - 4\,a\,b - x\,y + z - 3\,x\,y + 6\,a\,b + 7\,x\,y - 3\,z - 4\,a\,x - 6\,x\,y + a\,x$.

Ans. $2\,a\,b - 3\,x\,y - 2\,z$.

2. Add together $17\,ax^2 - 9\,a\,x\,y + 6\,z - 24 + 6 + 2\,a\,x\,y - 2\,z + a\,x^2 - 13 - 14\,a\,x^2 + 6\,a\,x\,y - 4\,z$.

Ans. $4\,a\,x^2 - a\,x\,y - 31$.

SUBTRACTION.

28. Subtraction is the process of taking one quantity from another, or it is finding the difference between two quantities.

29. The **Subtrahend** is the quantity to be subtracted.

30. The **Minuend** is the quantity from which the subtrahend is to be subtracted.

31. The **Remainder** is the result.

Let us subtract $14\,a$ from $26\,a$.

$$\begin{array}{rl}
26\,a & \textit{Minuend.} \\
14\,a & \textit{Subtrahend.} \\
\hline
12\,a & \textit{Remainder.}
\end{array}$$

This shows that the minuend is equal to the sum of the subtrahend and remainder.

We could write the above as follows:

$$26a - 14a = 12a.$$

32. In a certain time A made 24 dollars and B lost 8 dollars. What was the difference in the profits of A and B. Gain or profit is considered $+$, and loss $-$. To find the difference we must take -8 dollars from 24 dollars.

$$24 \text{ dollars} - (-8 \text{ dollars}) = 32 \text{ dollars}.$$

33. When studying negative quantities (Art. 22), we learned that the difference between a $+$ quantity and a $-$ quantity is equal to the sum of the quantities. The above shows that:

34. *Subtracting a positive quantity is the same as adding an equal negative quantity.*

Art. 32 shows that *subtracting a negative quantity is the same as adding an equal positive quantity.*

To subtract one quantity from another, change the sign of the subtrahend and add the result to the minuend.

Subtract $5a$ from $8a$.

SUBTRACTING.			ADDING.
$8a$	*Minuend*		$8a$
$5a$	*Subtrahend*	or	$-5a$
$3a$	*Remainder*		$3a$

35. To subtract one polynomial from another *change the sign of each term of the subtrahend and add the result to the minuend.*

From $11abc + 3x - 7y + 48$ take $7abc - 6x + 9y - 76$.

$$\begin{array}{r}
11abc + 3x - 7y + 48 \\
-7abc + 6x - 9y + 76 \\
\hline
4abc + 9x - 16y + 124
\end{array}$$

EXAMPLES FOR PRACTICE.

1. From $-347axy$ take $223axy$. Ans. $-570axy$.
2. Subtract $4m^2 - 6n^3 + 73x$ from $-m^2 - 8n^3 + 83x$. Ans. $-5m^2 - 2n^3 + 10x$.
3. Take $2x^3 - y^2$ from the sum of $x^3 - 2xy + 3y^2$ and $xy + 4y^2$. Ans. $-x^3 - xy + 8y^2$.

4. From the sum of $2a^3 - a^2b - 5ab^2$ and $3a^2b - 5ab^2 - 4b^3$ take the sum of $-2a^3 - 7a^2b - 6b^3$ and $-6ab^2 + 5b^3$.

Ans. $4a^3 + 9a^2b - 4ab^2 - 3b^3$.

5. Subtract the sum of $-5x^3 - 12x^2 + 3$ and $-3x^4 + 2x^2 - 5x$ from the sum of $x^4 - 1$ and $2x^3 - 7x - 10x^2$.

Ans. $4x^4 + 7x^3 - 2x - 4$.

USE OF PARENTHESES.

36. The parentheses, brackets, etc., are used so frequently in Algebra and Engineering that their use must be thoroughly understood.

Suppose we have the expression,

$$a - 2m + (3m - n + 4x).$$

We know that it means that the quantity $3m - n + 4x$ is to be *added* to $a - 2m$. Performing the addition we have

$$a - 2m + 3m - n + 4x.$$

37. It will be noted that the signs are the same as though no parenthesis had been used.

The expression,

$$a - 2m - (3m - n + 4x)$$

means that the quantity $3m - n + 4x$ is to be *subtracted* from $a - 2m$. Subtracting $3m - n + 4x$ is the same as adding $-3m + n - 4x$. We have as a result,

$$a - 2m - 3m + n - 4x.$$

38. By comparing this expression with that of Art. 37 we see that when we removed the parenthesis, we simply changed all the signs in the parenthesis. Similarly

$$a^2 - (2ab + c) = a^2 - 2ab - c.$$

39. We can then make the following statements:

If a parenthesis is preceded by a + sign, the parenthesis signs may be removed without changing the + or − signs of the terms within the parenthesis.

If a parenthesis is preceded by a − sign, the parenthesis signs may be removed if the sign of each term within the parenthesis is changed from + to − or from − to +.

40. The same rules apply to brackets, braces, vinculum, etc.

41. After removing the parenthesis similar terms should be united.

Simplify the expression,

$$4\,m^2\,n - (8\,m\,n^2 + 3\,m^2\,n - n^2).$$

$$4\,m^2\,n - 8\,m\,n^2 - 3\,m^2\,n + n^2$$

or, $$m^2\,n - 8\,m\,n^2 + n^2.$$

42. Parentheses are often enclosed by brackets, etc. In such cases the same rules are followed. It is better to remove first the signs from the innermost pair.

$$3\,a - [2\,a - (a + 2)].$$

Removing the parenthesis we have,

$$3\,a - [2\,a - a - 2].$$

Removing the brackets we have,

$$3\,a - 2\,a + a + 2.$$

Combining similar terms, we have,

$$2\,a + 2.$$

EXAMPLES FOR PRACTICE.

Simplify,

1. $x^3 - z^3 - (x^3 + y^3 - z^3)$ Ans. $-y^3$.
2. $a - c + d - (a + c - d) - \overline{d - c - a}.$ Ans. $a - c + d$.
3. $2\,a - \left\{ b - [3\,a - (2\,b - a)] \right\}.$ Ans. $6\,a - 3\,b$.
4. $5\,x - \left(4\,x - \left[-3\,x - \left\{ 2\,x - x - 1 \right\} \right]\right)$ Ans. $1 - 3\,x$.

43. To enclose any terms in parenthesis we observe the same principles.

44. Suppose we wish to enclose in parenthesis the last three terms of the following expression, and have the + sign precede the parenthesis,

$$x^3\,y + x^2\,y^2 - x\,y^3 + y^4.$$

$$x^3\,y + (x^2\,y^2 - x\,y^3 + y^4).$$

If the parenthesis is to be preceded by the — sign, the sign of each term thus enclosed must be changed.

$$x^3\,y + x^2\,y^2 - x\,y^3 + y^4$$

$$x^3\,y - (-x^2\,y^2 + x\,y^3 .- y^4).$$

45. Another example. To enclose the last three terms in parenthesis preceded by the — sign.

$$3x - 2y + 5z - 4x$$
$$3x - (2y - 5z + 4x).$$

In the first expression the sign of $2y$ is — and if it is placed within the parenthesis the sign must be changed to +. It is, however, omitted, as the plus sign is understood.

MULTIPLICATION.

46. Multiplication is a method of finding or expressing one quantity as many times as is indicated by the other or others.

In Arithmetic the product of 4×3 is the same as 3×4. Similarly $a \times b = b \times a$.

47. Monomials. First multiply together the *coefficients;* then annex the *letters*, giving to each letter an exponent equal to the *sum* of the exponents of the factors.

48. The sign of the product is + when the signs of the factors are alike and — when the signs are unlike.

49. Multiply $3ab^2$ by $6a^2b$.

Multiplying the coefficients we have 18 for a new coefficient; $a \times a^2 = a^3$ and $b \times b^2 = b^3$.

The product is $18a^3b^3$.

In multiplying the above we proceed as follows: $a \times a^2 = a^3$ because $a^2 = a \times a$ and $a \times a \times a = a^3$. Similarly, $b^2 = b \times b$ and $b \times b \times b = b^3$.

50. *In multiplying letters we add the exponents.* Thus $a^1 \times a^1 = a^2$. Again, $c^3 \times c^5 = c^8$. The same holds good if the exponents are letters. Thus,

$$a^m \times a^m = a^{m+m} = a^{2m}.$$

51. Another example. Multiply $5a^2b^nx$ by $3a^mb^3x^3$.

The new coefficient is $5 \times 3 = 15$.

$$a^2 \times a^m = a^{2+m}$$
$$b^n \times b^3 = b^{3+n}$$
$$x \times x^3 = x^4$$

The product is $15a^{2+m}b^{3+n}x^4$.

52. Multiply $5\,a\,b^2\,c$ by $-2\,b\,c^3$.

The new coefficient will be $5 \times -2 = -10$.

The product of the letters is $a\,b^3\,c^4$ and the result,

$$-10\,a\,b^3\,c^4.$$

Had the sign of the first quantity been $-$ the product would be $10\,a\,b^3\,c^4$, because $-5 \times -2 = +10$.

EXAMPLES FOR PRACTICE.

1. Multiply $8\,x^2\,y$ by $7\,a\,y$. Ans. $56\,a\,x^2\,y^2$.
2. Multiply $-3\,a^3\,b^m$ by $4\,b^n\,c$. Ans. $-12\,a^3\,b^{m+n}\,c$.
3. Multiply $-7\,a^3\,b^x\,c^4$ by $-6\,b^x\,c^e$. Ans. $42\,a^3\,b^{2x}\,c^{4+e}$.
4. Multiply $16\,m\,n\,x^3$ by $-a\,b\,c^2$. Ans. $-16\,a\,b\,c^2\,m\,n\,x^3$.

53. As any number of terms in a parenthesis is considered as a *monomial* the coefficient of the *parenthesis* is multiplied.

Multiply $4\,(a + b)$ by $3\,a$.

$3\,a \times 4 = 12\,a$ and annexing the quantity in parenthesis we have,

$$12\,a\,(a + b).$$

54. Polynomials. Suppose we wish to multiply $7 + 2$ by 4. Since multiplication is a short method of finding the sum of the repetitions of a quantity,

$$\begin{array}{l} 7 + 2 \\ 7 + 2 \\ 7 + 2 \\ 7 + 2 \\ \hline 28 + 8 = 36 \end{array} \qquad \text{or} \qquad \begin{array}{l} 7 + 2 \\ 4 \\ \hline 28 + 8 = 36 \end{array}$$

55. Thus we see that we obtain the answer if we multiply *both* terms of the polynomial by the monomial.

Multiply $3\,a + 5\,b$ by $8\,x$.

$$\begin{array}{l} 3\,a + 5\,b \\ 8\,x \\ \hline 24\,a\,x + 40\,b\,x. \end{array}$$

Multiply $23\,a\,b^2 - 5\,x\,y + 3\,a\,y - 1$ by $-7\,a\,x^2\,y$.

$$\begin{array}{l} \qquad 23\,a\,b^2 - 5\,x\,y + 3\,a\,y - 1 \\ -7\,a\,x^2\,y \\ \hline -161\,a^2\,b^2\,x^2\,y + 35\,a\,x^3\,y^2 - 21\,a^2\,x^2\,y^2 + 7\,a\,x^2\,y. \end{array}$$

56. If both factors, or in other words if both multiplier and multiplicand. are polynomials we proceed in the same way: *multiply each term of the multiplicand by each term of the multiplier and add the products.*

In performing multiplication of polynomials the signs are of utmost importance.

57. Example. Multiply $4\,a - 3\,b$ by $2\,a + 7\,b$.

$$\begin{array}{lll}
4\,a & -\ 3\,b & \\
2\,a & +\ 7\,b & \\
\hline
8\,a^2 & -\ 6\,a\,b & \\
 & +\,28\,a\,b & -\,21\,b^2 \\
\hline
8\,a^2 & +\,22\,a\,b & -\,21\,b^2
\end{array}$$

58. Let us take another example. Multiply $4\,a^2 + 2\,a\,b + b^2$ by $4\,a^2 - 2\,a\,b + b^2$.

$$\begin{array}{lllll}
4\,a^2 & +\,2\,a\,b & +\,b^2 & & \\
4\,a^2 & -\,2\,a\,b & +\,b^2 & & \\
\hline
16\,a^4 & +\,8\,a^3\,b & +\,4\,a^2\,b^2 & & \\
 & -\,8\,a^3\,b & -\,4\,a^2\,b^2 & -\,2\,a\,b^3 & \\
 & & +\,4\,a^2\,b^2 & +\,2\,a\,b^3 & +\,b^4 \\
\hline
16\,a^4 & & +\,4\,a^2\,b^2 & & +\,b^4
\end{array}$$

$16\,a^4 + 4\,a^2\,b^2 + b^4$ Ans.

59. If some of the terms are figures the same rules are followed. Multiply $1 - 5\,x - 3\,x^2 + 2\,x^3$ by $3\,x - 5$.

$$\begin{array}{lllll}
1 & -\,5\,x & -\,3\,x^2 & +\,2\,x^3 & \\
3\,x & -\,5 & & & \\
\hline
3\,x & -\,15\,x^2 & -\,9\,x^3 & +\,6\,x^4 & \\
25\,x & +\,15\,x^2 & -\,10\,x^3 & & -\,5 \\
\hline
28\,x & & -\,19\,x^3 & +\,6\,x^4 & -\,5
\end{array}$$

$28\,x - 19\,x^3 + 6\,x^4 - 5$. Ans.

60. In Algebra it is customary to arrange the terms in the answer, and also sometimes in the example, so that the first term will be + and the exponents will be either an ascending or descending scale. Thus in the above problem we should write the answer.

$$6\,x^4 - 19\,x^3 + 28\,x - 5.$$

61. If the product of three factors is to be found, first multiply together two of them and then multiply the product by the third.

62. Example: Multiply together, $a + b$, $a^2 - a\,b + b^2$, $a^3 - b^3$.

$$\begin{array}{l}
a^2 - a\,b + b^2 \\
a + b \\
\hline
a^3 - a^2 b + a\,b^2 \\
\quad + a^2 b - a\,b^2 + b^3 \\
\hline
a^3 \qquad\qquad\quad + b^3
\end{array}$$

Now multiply this product by $a^3 - b^3$.

$$\begin{array}{l}
a^3 + b^3 \\
a^3 - b^3 \\
\hline
a^6 + a^3 b^3 \\
\quad - a^3 b^3 - b^6 \\
\hline
a^6 \qquad\quad - b^6
\end{array}$$

$a^6 - b^6$. Ans.

63. The product of two or more quantities may be *indicated* by enclosing them in parenthesis. Thus the above example would be written.

$$(a + b)\,(a^2 - a\,b + b^2)\,(a^3 - b^3).$$

EXAMPLES FOR PRACTICE.

Multiply the following:

1. $x + y$ by $x - y$. Ans. $x^2 - y^2$.
2. $2\,x - 3$ by $3\,x + 2$. Ans. $6\,x^2 - 5\,x - 6$.
3. $6\,m - 4$ by $3\,m - 2$. Ans. $18\,m^2 - 24\,m + 8$.
4. $3\,a^2 - a - 2$ by $3\,a + 2$. Ans. $9\,a^3 + 3\,a^2 - 8\,a - 4$.
5. $2\,m^2 + 9\,m - 5$ by $3\,m^2 - 7\,m + 4$.
Ans. $6\,m^4 + 13\,m^3 - 70\,m^2 + 71\,m - 20$.
6. Find the product
$(3\,a^2 - a - 2)\,(a + 1)\,(3\,a - 2)$.
Ans. $9\,a^4 - 13\,a^2 + 4$.

DIVISION.

64. The process of finding one of two factors when the product and one factor are known is called Division. Thus we see that division is the converse of Multiplication.

65. The *product* is called the **Dividend.**

66. The *given factor* is called the **Divisor.**

67. The *result* or the other factor is called the **Quotient.**

68. As in multiplication, like signs give + and unlike signs —.

If the divisor and dividend are both + the quotient is +.
If the divisor and dividend are both — the quotient is +.
If the divisor is + and the dividend — the quotient is —.
If the divisor is — and the dividend + the quotient is —.

69. Divide $16\,a\,b$ by $8\,a$.

$$\frac{16\,a\,b}{8\,a} = 2\,b.$$

Monomials. To find the coefficient of the quotient, divide the coefficient of the dividend by the coefficient of the divisor.

70. To find the exponent of any letter of the quotient, *subtract* the exponent of the letter of the divisor from the exponent of the letter of the dividend.

71. Divide $27\,a^3\,b^3$ by $3\,a^2\,b$.

To find the coefficient of the quotient we divide 27 by 3. $27 \div 3 = 9$.

Now we subtract the exponent of a^2 from that of a^3 and obtain 1 as a result because $3 - 2 = 1$, and any quantity with the exponent 1 is the quantity itself. $a^1 = a$.

Subtracting the exponent of b from that of b^3 we get 2 because $3 - 1 = 2$. Hence the exponent of b should be 2. The entire quotient is $9\,a\,b^2$.

72. In Multiplication we *add* the exponents and in Division we *subtract* them.

Divide a^5 by a^2.

$$\frac{a^5}{a^2} = a^3 \text{ ; because } 5 - 2 = 3.$$

$$\frac{a^m}{a^n} = a^{m-n}.$$

73. Another example. Divide $36\,x^2 y\,z^3$ by $-\,4\,x\,y\,z$.

$$\frac{36\,x^2 y\,z^3}{-\,4\,x\,y\,z} = -\,9\,x\,z^2.$$

In this case the sign of the quotient is — because the signs of the dividend and divisor are unlike. The exponents of y are the same, hence they cancel.

74. The process of dividing a polynomial by a monomial is very simple. Consider the polynomial made up of several monomials; divide each monomial by the divisor. In doing this pay careful attention to the signs, remembering that like signs produce + and unlike —.

75. Divide $12 x^4 y^6 - 16 x^5 y^5 + 20 x^6 y^4 - 28 x^7 y^8$ by $4 x^4 y^3$.

$$\begin{array}{r|l} 4 x^4 y^3 & 12 x^4 y^6 - 16 x^5 y^5 + 20 x^6 y^4 - 28 x^7 y^8 \\ \hline & 3 y^3 - 4 x y^2 + 5 x^2 y - 7 x^3 y^5 \end{array}$$

76. Another example: Divide $15 a^m b^n c^r - 35 a^{m+2} b^{2n} c$ by $5 a^m b^n c$.

$$\begin{array}{r|l} 5 a^m b^n c & 15 a^m b^n c^r - 35 a^{m+2} b^{2n} c. \\ \hline & 3 c^{r-1} - 7 a^2 b^n. \end{array}$$

EXAMPLES FOR PRACTICE.

1. Divide $20 x^4 y z + 15 x y z^3 - 10 x^2 y$ by $-5 x y$.
   Ans. $-4 x^3 z - 3 z^3 + 2 x$.
2. Divide $9 a^4 + 27 a^3 - 15 a^2 - 6 a$ by $3 a$.
   Ans. $3 a^3 + 9 a^2 - 5 a - 2$.
3. Divide $-75 m^4 n + 30 m^3 n^2 - 15 m^2 n$ by $15 m^2 n$.
   Ans. $-5 m^2 + 2 m n - 1$.

77. Polynomials. We have seen that the division of monomials is similar to short division in Arithmetic. The division of polynomials is like long division in Arithmetic.

78. Let us take an example. Divide $a^3 - 3 a^2 y + 3 a y^2 - y^3$ by $a^2 - 2 a y + y^2$.

First we should write the divisor and dividend as in long division in Arithmetic.

$$\begin{array}{r|l} a^2 - 2 a y + y^2 & a^3 - 3 a^2 y + 3 a y^2 - y^3 \, (a - y \\ & a^3 - 2 a^2 y + a y^2 \\ \hline & \quad - a^2 y + 2 a y^2 - y^3 \\ & \quad - a^2 y + 2 a y^2 - y^3 \\ \hline \end{array}$$

To find the first term of the quotient divide the a^3 by a^2. $\frac{a^3}{a^2} = a$. Hence a is first term. Now we multiply each term in the divisor by a and after placing the results under similar terms of the dividend we subtract. We then divide $-a^2y$ by a^2 and get $-y$ for the next term. Multiplying again by $-y$ we find that the terms are the same as those brought down.

79. Let us take another example. Divide $2a^3b - 2ab^3$ by $ab - b^2$.

$$\begin{array}{l}
ab - b^2\ )\ 2a^3b - 2ab^3\ (\ 2a^2 + 2ab \\
\qquad\qquad \underline{2a^3b - 2a^2b^2} \\
\qquad\qquad 2a^2b^2 - 2ab^3 \\
\qquad\qquad \underline{2a^2b^2 - 2ab^3}
\end{array}$$

As before we divide $2a^3b$ by ab and obtain $2a^2$ for the first term of the quotient. Now multiplying both terms of the divisor by $2a^2$ we get $2a^3b - 2a^2b^2$. The $2a^3b$ cancels but we have no quantity similar to $-2a^2b^2$ so we bring it down *with the sign changed.* The new dividend is $2a^2b^2 - 2ab^3$ and we find that the divisor is contained in it $2ab$ times.

80. Divide $m^4 - 1$ by $m - 1$.

$$\begin{array}{l}
m - 1\ )\ m^4 - 1\ \ (\ m^3 + m^2 + m + 1.\ \text{Ans.} \\
\qquad\quad \underline{m^4 - m^3} \\
\qquad\qquad + m^3 - 1 \\
\qquad\qquad \underline{+ m^3 - m^2} \\
\qquad\qquad\qquad + m^2 - 1 \\
\qquad\qquad\qquad \underline{+ m^2 - m} \\
\qquad\qquad\qquad\qquad + m - 1 \\
\qquad\qquad\qquad\qquad \underline{+ m - 1}
\end{array}$$

If the dividend is not exactly divisable by the divisor, the remainder should be placed over the divisor and connected to the quotient by the proper sign.

EXAMPLES FOR PRACTICE.

1. Divide $a^4 + 4c^4$ by $a^2 - 2ac + 2c^2$.
   Ans. $a^2 + 2ac + 2c^2$.
2. Divide $x^4 + 4x^2y^2 + 3y^4$ by $x + 2y$.
   Ans. $x^3 - 2x^2y + 8xy^2 - 16y^3 + \dfrac{35y^4}{x + 2y}$.
3. Divide $10a^3 - 20a^2x + 30x^3$ by $a - x$.
   Ans. $10a^2 - 30ax + 30x^2$.
4. Divide $2c^4 - 3c^3 - 12 - 4c^2 - 11c$ by $2c^2 + c - 4$
   Ans. $c^2 - 2c - 3$.

CANCELLATION.

As shown in Arithmetic, cancellation often shortens the work and a few rules concerning its use will now be given.

81. CASE I. Suppose we have the expression

$$\frac{18 \times 6 \times 10 \times 2}{24 \times 3} = ?$$

We can perform the cancellation thus,

$$\frac{\overset{6}{\cancel{18}} \times \overset{1}{\cancel{6}} \times \overset{5}{\cancel{10}} \times \overset{1}{\cancel{2}}}{\underset{\underset{\cancel{2}_1}{\cancel{4}}}{\cancel{24}} \times \cancel{3}_1} = 30$$

82. If the signs in the numerator had been + instead of × we could not have cancelled as above. When we cancel we simply divide out factors. Now if these quantities are connected by the plus sign, we cannot cancel out a factor unless it is contained a whole number of times *in every quantity so connected.*

For instance let us take the expression:

$$\frac{18 + 6 + 10 + 2}{24 \times 3} = ?$$

83. In the numerator, 2 is contained a whole number of times in each quantity. Also, 2 is a factor of one quantity, (24), of the denominator. Hence we divide *each* quantity of the numerator and one quantity of the denominator by 2. Thus,

$$\frac{\overset{9}{\cancel{18}} + \overset{3}{\cancel{6}} + \overset{5}{\cancel{10}} + \overset{1}{\cancel{2}}}{\underset{12}{\cancel{24}} \times 3} = \frac{9 + 3 + 5 + 1}{12 \times 3} = \frac{18}{36} = \frac{1}{2}$$

84. Hence we may say, when the quantities, (letters or numbers) of the numerator are connected by the + (or —) sign, and those in the denominator by the × sign: *Cancel the same factor from each quantity of the numerator as from one quantity of the denominator.*

85. CASE II. Suppose we have the expression,

$$\frac{18 \times 6 \times 10 \times 2}{24 + 3} = ?$$

We see that this case is exactly the reverse of the preceding.

The factor 3 is contained in both 24 and 3 of the denominator and in 18 of the numerator. We cancel as follows:

$$\frac{\overset{6}{\cancel{18}} \times 6 \times 10 \times 2}{\underset{8}{\cancel{24}} + \underset{1}{\cancel{3}}} = \frac{6 \times 6 \times 10 \times 2}{8 + 1} = \frac{720}{9} = 80.$$

86. In regard to this case we may say; when quantities in the numerator are connected by the × sign and those of the denominator by the + or — sign: *Cancel the same factor from each quantity in the denominator as from one quantity of the numerator.*

87. CASE III. Again, suppose all the signs are either + or —.

$$\frac{6 - 9 + 81 - 3}{27 + 18} = ?$$

First, we must see if there is some factor contained in each quantity. We see that 3 is the factor. Then we can cancel as follows:

$$\frac{\overset{2}{\cancel{6}} - \overset{3}{\cancel{9}} + \overset{27}{\cancel{81}} - \overset{1}{\cancel{3}}}{\underset{9}{\cancel{27}} + \underset{6}{\cancel{18}}} = \frac{2 - 3 + 27 - 1}{9 + 6} = \frac{25}{15} = \frac{5}{3}$$

88. When the quantities in both the numerator and denominator are connected by the + (or —) sign : *Cancel the factor or factors which are contained in every quantity.*

89. We may have examples containing both letters and figures but the principles governing the cancellation are the same, that is, if letters occur in both the numerator and the denominator they may be cancelled in the same manner as figures.

Let us consider the expression,

$$\frac{16 + 3 + 2 - 15}{3 + 2 + 5} = ?$$

90. We cannot cancel because there is no factor (except 1) which is contained without a remainder in every quantity. However, in the following expression cancellation greatly simplifies the work.

$$\frac{\overset{1}{\cancel{2}} \times \overset{1}{\cancel{3}} \times \overset{1\,\cancel{2}}{\cancel{22}} \times \overset{1\,\cancel{8}}{\cancel{40}}}{\underset{1}{\cancel{11}} \times \underset{1}{\cancel{5}} \times \underset{1}{\cancel{2}} \times \underset{\cancel{2}_1}{\cancel{16}} \times \cancel{8}_1} = \frac{1}{1} = 1$$

EQUATIONS.

91. When two expressions are equal to each other, they may be written with the sign of equality between them. The whole expression is called an **equation.** Thus,

$$a + 2b + x = m + 4n.$$

92. The *first member* is the expression on the left of the sign of equality and the *second member* is the expression on the right. The members are also called *sides.*

93. In the above equation $a + 2b + x$ is the first member, and $m + 4n$ the second. If we should change any quantity such as a to the *other side* it would mean that a is written on the right-hand side instead of on the left-hand side.

94. Since an equation is an expression of **equality,** the sum of the quantities on the left-hand side of the sign of equality is equal to the sum of those on the right-hand side. In other words, the first member is equal to the second.

In the equation,

$$a + 2b = m + n$$

let $a = 3$, $b = 2$, $m = 6$ and $n = 1$.

Substituting these values we have,

$$\begin{aligned} a + 2b &= m + n \\ 3 + (2 \times 2) &= 6 + 1 \\ 3 + 4 &= 6 + 1 \\ 7 &= 7 \end{aligned}$$

Suppose now we add 8 to each side of the equation; the result is,

$$\begin{aligned} a + 2b + 8 &= m + n + 8 \\ 3 + 4 + 8 &= 6 + 1 + 8 \\ 15 &= 15 \end{aligned}$$

We see that the first member is still equal to the second. Now let us divide each side by a which equals 3.

$$\frac{a + 2b + 8}{a} = \frac{m + n + 8}{a}$$

$$\frac{3 + 4 + 8}{3} = \frac{6 + 1 + 8}{3}$$

$$\frac{15}{3} = \frac{15}{3} \text{ or } 5 = 5$$

95. Thus we see that *if the same quantity is* **added** *to both sides, or if both sides are* **divided** *by the same quantity, the first member is still equal to the second.* Similarly, it can be shown that if the same quantity is **subtracted** from *both sides* or if both sides are **multiplied** by the same quantity, the left side of the equation is equal to the right.

96. Suppose, $a = 6$, $b = 4$ and $c = 2$, and we wish to find the value of the expression,

$$a + b - 3\,c.$$

In such cases we write it as an equation with the unknown quantity represented by x. Thus,

$$x = a + b - 3\,c.$$

The expression means that x is equal to $a + b - 3\,c$.

Then,

$$x = a + b - 3\,c.$$
$$x = 6 + 4 - (3 \times 2)$$
$$x = 6 + 4 - 6$$
$$x = 4$$

We have found the value of x or in other words we have *solved the equation.*

97. Another example: Find the value of x from the equation.

$$x = \frac{6\,a + 5\,c}{2\,b}, \text{ if } a = 3, b = 5, \text{ and } c = 2.$$

Replacing a, b and c, by their respective values,

$$x = \frac{(6 \times 3) + (5 \times 2)}{2 \times 5}$$

$$x = \frac{18 + 10}{10}$$

$$x = 2.8 \text{ Ans.}$$

98. Suppose an engine makes 125 revolutions per minute and develops .4 horse-power per revolution, what is the total horse-power?

As the total horse-power is the unknown quantity, we will let it be represented by x. The known quantities, 125 revolutions and .4 horse-power, can be represented by a and b respectively. Thus,

$a = 125$ revolutions. $b = .4$ horse-power.

$x =$ the total horse-power.

We know that the total horse-power is equal to the number of revolutions multiplied by the horse-power per revolution, or,

$$x = a \times b.$$
$$x = 125 \times .4.$$
$$x = 50 \text{ horse-power.} \quad \text{Ans.}$$

99. Equations are not always written with x representing the unknown quantity, any other letter may be used.

TRANSPOSING.

100. There are many equations in which the unknown quantity is not alone on the left-hand side; it frequently happens that it is among several known quantities on the right-hand side. In such cases it is usual to *rewrite* the equation so that the unknown quantity will be alone on the left-hand side. To do this quantities must be **transposed** from one side to the other.

101. Let us consider this equation,

$$a = c - x.$$

If we subtract a from both sides we will have,

$$a - a = c - a - x.$$

Now let us add x to both sides,

$$a - a + x = c - a - x + x.$$

We know that $a - a = 0$ and $-x$ and $+x$ cancel, hence we may write;

$$x = c - a.$$

102. In brief, we have *transposed* the x to the left-hand side, *transposed* the a to the right-hand side (because a known quantity) and let the *c remain on the right-hand side* (because a known quantity).

103. It will be noted that in changing the x to the other side, the **sign is changed.** Also the sign of the a is now $-$ instead of $+$. The sign before c is the same as before.

104. Any term may be transposed from one side to the other by changing the sign.

105. Let us consider this equation,

$$5x + 3ab = 2x + 3ab + m.$$

Here we have terms containing x on both sides. First we place the $2x$ on the left-hand side *with its sign changed.*

$$5x + 3ab - 2x = 3ab + m.$$

Next we write the 3 $a\,b$ on the right-hand side *with its sign changed.*

$$5x - 2x = 3ab - 3ab + m.$$

Combining like terms,

$$3x = m.$$

Since $3x = m$, one x equals $m \div 3$ or $x = \frac{m}{3}$

106. In other words *to find the value of x we divide by the* **coefficient** *of x.*

NOTE. In solving problems, be careful to make a distinction between the letter x and the multiplication sign ×.

EXAMPLES FOR PRACTICE.

1. Find the value of x in the equation, $8x - 12 = 6x + 4$.
Ans. $x = 8$.

2. Solve the equation, $37x - (4 + 7) = 41x + 25$.
Ans. $x = -9$.

3. Find the value of x in the equation,
$2ax + c = 3c - 5ax + 26d$
$a = 3$, $c = 8$ and $d = 1$. Ans. $x = 2$.

4. Solve the equation,
$7x - ab + c = 3x + 12$.
$a = 5$, $b = 3$ and $c = 7$. Ans. $x = 5$.

107. Let us consider another equation:

$$\frac{x}{a} + c = a^2 - b^2.$$

First we transpose c to the right-hand side.

$$\frac{x}{a} = a^2 - b^2 - c.$$

To obtain x alone, we must multiply both sides by a.

$$\frac{x}{a} \times a = a\,(a^2 - b^2 - c).$$

or, $$x = a^3 - a\,b^2 - a\,c.$$

108. The left-hand side of the equation shows that multiplying a fraction by the *denominator* gives the numerator as a result, that is, $\frac{x}{a} \times a = x$. Similarly if we have the fraction $\frac{3}{8}$ and multiply it by 8 we get the numerator 3 as a result.

109. We find the value of x from the equation $\frac{3x}{4} = 9$ by multiplying both sides by 4, and then dividing by the coefficient of x.

$$3x = 9 \times 4 = 36,$$
$$3x = 36$$
$$x = \frac{36}{3} = 12.$$

110. Consider the equation,

$$a + 2ax - c = 36 - 3ab.$$

First, we place a and c on the right side with the signs changed, that is we transpose the terms.

$$2ax = 36 - 3ab - a + c.$$

From Art. 11 it is evident that $2a$ is the coefficient of x; hence we divide both sides by $2a$.

$$x = \frac{36 - 3ab - a + c}{2a}.$$

111. Stated in words: The terms on the left-hand side which do not contain the unknown quantity should be placed on the right-hand side with the signs changed and all quantities in the same term with the unknown quantity should be transposed by multiplying or dividing both sides of the equation by them.

112. Find the value of P from the following formula:

$$P = \sqrt[3]{\frac{ABC}{1.4\left(B + \frac{A^2}{C^2}\right)}}, \text{ when } A = 11,\ B = 1.71, \text{ and } C = 1.3.$$

Solution:

$$P = \sqrt[3]{\frac{11 \times 1.71 \times 1.3}{1.4\left(1.71 + \frac{121}{1.69}\right)}}$$
$$= \sqrt[3]{\frac{24.453}{1.4 \times 73.308}}$$
$$= \sqrt[3]{.238} = .619 + \text{Ans.}$$

113. In solving equations remember the rule:

Terms containing unknown quantities may be transposed to the left-hand side of the equation by changing the signs. ***Terms con-***

taining known quantities may be transposed to the right-hand side by changing the signs.

To find the value of the unknown quantity divide the right-hand side by the coefficient of the unknown quantity. It is readily seen that equations may be solved if the unknown quantity is on the right side. Thus $2 = x$ is the same as $x = 2$. It is better to leave the unknown quantity on the right side, if changing it necessitates many changes in the known quantities.

EXAMPLES FOR PRACTICE.

1. What is the value of x in the equation,

$\frac{x}{3} + 2\,(3 + 4) = 6 - 3 + 24$? Ans. $x = 39$

2. Find the value of x from the equation,

$6\,ax + 3\,a^2 = 3\,a - 3\,ab$ Ans. $x = \frac{1 - b - a}{2}$

3. What is the value of x in the equation,

$3\,a + 2\,b - x = 6$? Let $a = 2$ and $b = 1$ Ans. $x = 2$

4. What is the value of x in the following equation, where $a = 3$, $b = 2$, and $c = 1$?

$2\,x - b = 20 + x - 3\,a\,c$ Ans. $x = 13$

5. If $a = 5$, $b = 2$, and $c = 3$, what is the value of the equation,

$x + 4\,ax = 21 + 2\,c + \frac{9\,a\,b}{2\,c}$? Ans. $x = 2$

114. Suppose we have the equation,

$$H = \frac{PLAN}{33{,}000},$$

in which H = 50 horse-power, P = 49.12 pounds, L = $1\frac{1}{2}$ feet, and A = 113.1 square inches. The value of N is unknown but may be found by solving this equation.

We know that if $H = \frac{PLAN}{33{,}000}$, we can write it thus:

$$33{,}000\ H = PLAN \text{ or } PLAN = 33{,}000\ H.$$

This is the same as multiplying both sides by 33,000.

Now we may consider PLA the coefficient of N;

$$N = \frac{33000\ H}{PLA}$$

In other words we have placed the PLA on the other side. By substituting the values given above we may solve for N.

$$\begin{aligned} N &= \frac{33000\ H}{PLA} \\ &= \frac{33000 \times 50}{49.12 \times \frac{3}{2} \times 113.1} \\ &= 198 + \end{aligned}$$

115. Suppose we wish to find the value of a in the following equation when $b = 2$, $c = 3$, $d = 4$, and $e = 5$.

$$6\,b\,c = a + \sqrt{d + e}$$

We can re-write the equation in order to get a upon the left side, thus,

$$\begin{aligned} a + \sqrt{d + e} &= 6bc \\ a &= 6bc - \sqrt{d + e} \\ a &= 6 \times 2 \times 3 - \sqrt{4 + 5} \\ &= 36 - \sqrt{9} \\ &= 36 - 3 \\ &= 33 \end{aligned}$$

We place the $\sqrt{d + e}$ on the right-hand side of the equation with the sign changed. If the sign before the radical had been $-$ instead of $+$ we would have changed it to plus, thus,

$$a = 6\,b\,c + \sqrt{d + e}$$

When solving examples it often gives a more accurate answer to simply indicate operations instead of actually performing them.

116. Suppose we wish to obtain the value of the expression,

$$\sqrt{(\sqrt{890})^2 + 10}.$$

We might perform the work by first extracting the square root of 890, which is approximately 29.83. The above expression would then become,

$$\sqrt{(29.83)^2 + 10}$$

If now we continue the operations as indicated, we shall have,

$$\sqrt{889.8289 + 10}$$

$$\sqrt{899.8289}$$

$$29.99 +$$

117. Now suppose we take the same expression, $\sqrt{(\sqrt{890})^2 + 10}$ and simplify it in another way.

We will not extract the square root of the quantity 890 as before, because the *square of the square root* of 890, is 890. Hence, for the quantity $(\sqrt{890})^2$ we may write 890. The expression then becomes $\sqrt{890 + 10}$ or $\sqrt{900}$, the value of which is 30.

It is seen that by one method we get 29.99 + as the result and by the other method we get 30. The latter result is the exact one, and at the same time the operation is more simple.

118. Suppose we have the equation,

$$W = .0033 \times 10^{-7} \times n$$

We can write it thus,

$$W = .0033 \times \frac{1}{10^7} \times n$$

Now, any quantity having an exponent preceded by the minus sign, as a^{-2} can be written as a fraction. The quantity is placed in the denominator with the sign of the exponent changed; the numerator being 1. Thus,

$$a^{-2} = \frac{1}{a^2}; \quad x^{-6} = \frac{1}{x^6}; \quad 10^{-8} = \frac{1}{10^8} = \frac{1}{100{,}000{,}000}$$

We can write the above equation,

$$W = .0033 \times \frac{1}{10{,}000{,}000} \times n.$$

If the value of n is known the equation can be readily solved.

EXAMPLES FOR PRACTICE.

Solve the following equations:

1. $35x - 3(4 + 6) = 30x - 5.$ Ans. $x = 5.$
2. $6x - (6 + 3x) = 12 + x - 14.$ Ans. $x = 2.$
3. Given the equation $3a - 2bx = 14 + 3a - (4c - 22).$
   $a = 4$, $b = 3$ and $c = 3.$
   Ans. $x = -4.$
4. Given the equation $6x - \frac{b}{a^{-4}} + c = 4ab + 4c.$
   $a = 2$, $b = 5$ and $c = 4.$
   Ans. $x = 22.$

MENSURATION.

119. Mensuration is the branch of mathematics which treats of the measurement of geometrical figures; that is, measurements of length, area and volume.

PRELIMINARY DEFINITIONS.

LINES.

120. A *line* has length only; it is produced by the motion of a point.

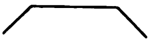
121. A *straight line* is one that has the same direction throughout. It is the shortest distance between two points.

122. A *curved line* is one that is constantly changing in direction; it is sometimes called a curve.

123. A *broken line* is one made up of several straight lines.

STRAIGHT LINE.

CURVED LINE.



BROKEN LINE.

PARALLEL LINES.

124. *Parallel lines* are lines which lie in the same plane and are equally distant from each other at all points. They cannot meet, however far they may be produced (that is, extended in either direction.)

125. A *horizontal line* is one having the direction of a line drawn upon the surface of water that is at rest. It is a line parallel to the horizon or sea level.

126. A *vertical line* is one that has the direction of a thread suspended from its upper end and having a weight at its lower end. It is a line perpendicular to a horizontal line.

127. An *oblique line* is a line that is neither vertical nor horizontal.

128. In order to distinguish between lines, they are *lettered.* Thus if one end of a line is marked *a* and the other end *b* we call it "the line *a b*" or "the line *b a*."

SURFACES.

129. A surface has two dimensions; ***length*** and ***breadth***

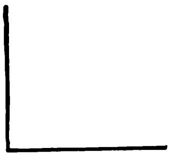
130. A *plane surface* is perfectly flat. A straight edge placed in any direction on a plane surface will be in contact at every point.

131. A *curved surface* is a surface no part of which is plane.

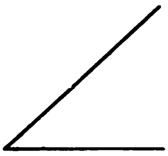
132. A *plane figure* is a combination of lines all of which lie in a plane surface.

ANGLES.

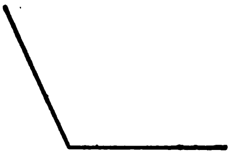
133. An angle is the difference in direction of two lines. The lines are called the *sides* and the point of meeting is called the *vertex*. The size of an angle depends upon the amount of divergence of the sides and is independent of the length of these lines.



RIGHT ANGLE.



ACUTE ANGLE.



OBTUSE ANGLE.

134. If one straight line meets another (at some point not an extremity) and the two angles thus formed are equal they are *right angles* and the lines are perpendicular to each other.

135. An *acute angle* is less than a right angle.

136. An *obtuse angle* is greater than a right angle.

137. Acute and obtuse angles are sometimes called *oblique angles*.

138. Angles are distinguished by placing letters on the lines forming the sides and at the vertex. Suppose the letter A is placed at the vertex and the letters B and C on the sides; we would read the angle as "the angle B A C" or "the angle C A B." The letter representing the vertex must be placed between those representing the sides. If but one letter is used it should be placed at the vertex and the angle is then designated as "the angle whose vertex is A" or simply "the angle A." In writing, the sign $\angle$ is often used in place of the word "angle."

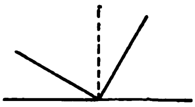
139. As *two right angles* are formed when one line meets another (see Art. 134), we know that

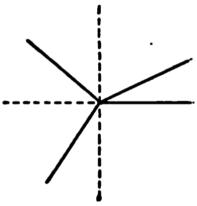
The sum of all the angles about a point on one side of a straight line is equal to two right angles.

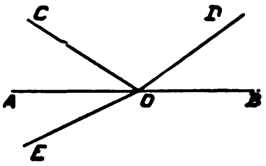
140. It also follows that the sum of all the angles about any point is equal to **four right angles.**

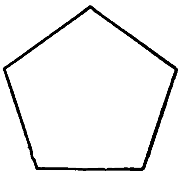
141. Consider the angles about the point O; angle A O C + angle C O D + angle D O B is equal to two right angles. Angle A O E + angle E O B is equal to two right angles.

∠ A O C + ∠ C O D + ∠ D O B + ∠ A O E + ∠ E O B = **four right angles.**

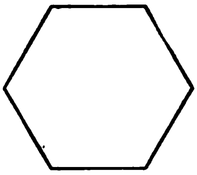




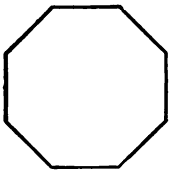




PENTAGON.

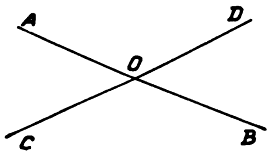


HEXAGON.



OCTAGON.

142. When two lines intersect they form four angles. The angles A O C and A O D are called **adjacent** angles because they have the same vertex O and the line A O is common, that is, it forms a side of each angle. The angles A O C and D O B are opposite angles.



From Art. 139 it follows that

∠ A O C + ∠ A O D = two right angles.

It may also be proved that ∠ A O C = ∠ D O B.

POLYGONS.

143. A polygon is a plane figure bounded by straight lines.

144. The boundary lines are called the ***sides*** and the sum of the sides is called the ***perimeter.***

145. Polygons are classified according to the number of sides.

A *triangle* is a polygon of *three* sides.

A *quadrilateral* is a polygon of *four* sides.

A *pentagon* is a polygon of *five* sides.

A *hexagon* is a polygon of *six* sides.

A *heptagon* is a polygon of *seven* sides.

An *octagon* is a polygon of *eight* sides.

A *decagon* is a polygon of *ten* sides.

A *dodecagon* is a polygon of *twelve* sides.

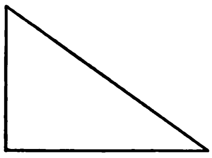
An *equilateral* polygon is one all of whose sides are equal.

An *equiangular* polygon is one all of whose angles are equal.

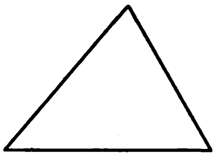
A *regular polygon* is one all of whose angles are equal and all of whose sides are equal.

TRIANGLES.

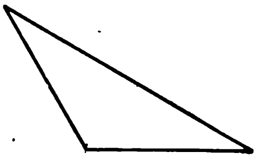
146. A *triangle* is a figure enclosed by three straight lines. The bounding lines are the *sides*, and the points of intersection of the sides are the *vertices*. The *angles* of a triangle are the angles formed by the sides.



RIGHT-ANGLED TRIANGLE.



ACUTE-ANGLED TRIANGLE.



OBTUSE-ANGLED TRIANGLE.

147. A *right-angled* triangle, often called a *right* triangle, is one that has a right angle. The longest side (the one opposite the right angle) is called the hypothenuse and the other lines are called sides or legs.

148. An *acute-angled* triangle is one that has all of its angles acute.

149. An *obtuse-angled* triangle is one that has an obtuse angle.

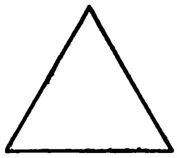
150. In an *equilateral* triangle all of the sides are equal.

151. If all of the *angles* of a triangle are equal, the figure is called an *equiangular* triangle.

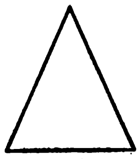
152. A triangle is called *scalene* when no two of its sides are equal.

153. In an *isosceles* triangle two of the sides are equal.

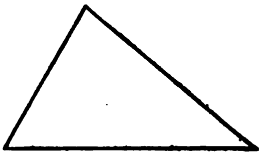
154. The *base* of a triangle is the lowest side; it is the side upon which the triangle is supposed to stand. Any side may, however, be taken as the base. In an isosceles triangle, the side which is not one of the equal sides is usually considered as the base.



EQUILATERAL TRIANGLE.

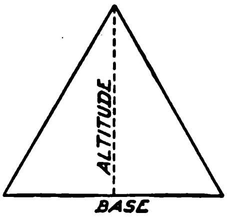


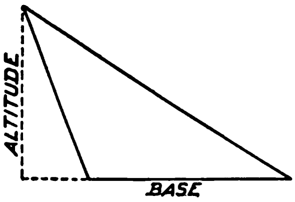
ISOSCELES TRIANGLE.

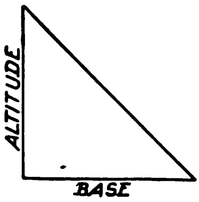


SCALENE TRIANGLE.

155. The *altitude* of a triangle is the perpendicular drawn from the vertex to the base. In some triangles it is necessary to produce the base so that the altitude may meet it. In a right triangle one leg may be considered as the base and the other the altitude.







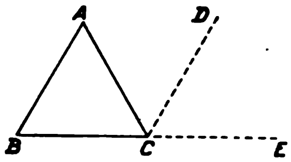


Fig. 1.

156. Let us consider Fig. 1. First produce (extend) B C to any point such as E and draw C D parallel to B A. It may be proved that angle A B C is equal to angle D C E and B A C equal to A C D. The angle A C B is of course equal to itself. Adding we have,

$$ABC + BAC + ACB = DCE + ACD + ACB$$

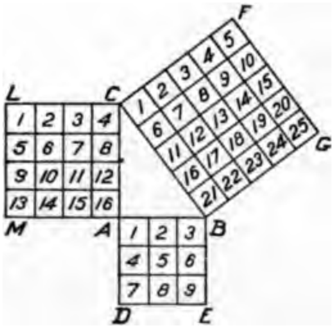
But since the sum of the angles about a point on one side of a straight line is equal to two right angles (see Art. 139) we have,

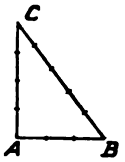
$$DCE + ACD + ACB = \text{two right angles};$$
$$\text{therefore } ABC + BAC + ACB = \text{two right angles}.$$

157. Hence we may say that *the sum of all the angles of any triangle is equal to two right angles.*

158. The Right Triangle. If in the right triangle A B C the side A B is 3 inches long, and the side A C is four inches long the side B C (the hypothenuse) is 5 inches long.

By considering the accompanying figure we find that the square A B E D constructed on A B contains 9 squares. The





square L C A M constructed on A C contains 16 squares and the square C F G B constructed on B C contains 25 squares. Now,

$$9 + 16 = 25$$
$$3^2 + 4^2 = 5^2$$
$$\overline{AB}^2 + \overline{AC}^2 = \overline{BC}^2$$

159. Then we may say: *The square of the hypothenuse is equal to the sum of the squares of the other two sides.* This statement is true for **all right-angled triangles**; it does not, however, apply to triangles other than right.

This principle is very useful for finding the lengths of various lines.

Example. A right triangle has sides 6 inches and 10 inches long respectively; what is the length of the hypothenuse?

$$6^2 + 10^2 = \text{square of hypothenuse.}$$
$$36 + 100 = 136$$
$$\sqrt{136} = 11.66 + \text{inches. Ans.}$$

160. Suppose we know the length of the hypothenuse and of one short side. We find the length of the other short side by means of the above.

Example. The hypothenuse of a right triangle is 24 feet long and one of the short sides is 9 feet. How long is the other short side?

$24^2 - 9^2 =$ the square of the required side.
$576 - 81 = 495$
$\sqrt{495} = 22.248 +$ feet. Ans.

Example. One side of a right triangle is 15 inches; if the hypothenuse is 21 inches, what is the other side?

$21^2 - 15^2 =$ the square of the other side.
$441 - 225 = 216$
$\sqrt{216} = 14.696 +$ inches. Ans.

EXAMPLES FOR PRACTICE.

1. One side of a right triangle is 8 feet and the other is 9 feet; what is the length of the hypothenuse?

Ans. 12 feet (about).

2. The hypothenuse of a right triangle is 18 inches and one side is 3 inches; what is the other side?

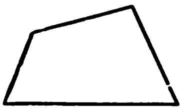
Ans. 17.748 + inches.

3. The two sides of a right triangle are 16 and 21 feet respectively. Find the length of the hypothenuse.

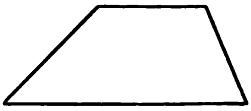
Ans. 26.4 feet.

4. The hypothenuse of a right triangle is 23. What is the length of the other two sides if they are equal?

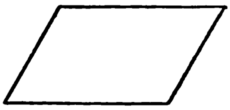
Ans. 16.263.



QUADRILATERAL.



TRAPEZOID.



PARALLELOGRAM.

QUADRILATERALS.

161. A *quadrilateral* is a plane figure bounded by four straight lines.

The *diagonal* of a quadrilateral is a straight line joining two opposite vertices.

162. A *trapezium* is a quadrilateral, no two of whose sides are parallel.

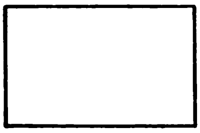
163. A *trapezoid* is a quadrilateral having two sides parallel.

The parallel sides are called the *bases* and the perpendicular distance between the bases is called the *altitude.*

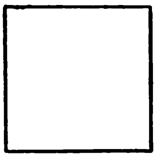
164. A *parallelogram* is a quadrilateral whose opposite sides are parallel.

The *altitude* of a parallelogram is the perpendicular distance between the *bases*, which are the parallel sides.

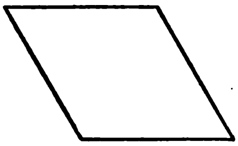
A diagonal of a parallelogram divides the parallelogram into *two equal triangles.*



RECTANGLE.



SQUARE.



RHOMBUS.

There are four kinds of parallelograms.

165. A *rectangle* is a parallelogram whose angles are right angles.

166. A *square* is a rectangle, all of whose sides are equal.

167. A *rhombus* is a parallelogram which has four equal sides; the angles, however, are not right angles.

168. A *rhomboid* is a parallelogram whose adjacent sides are unequal; the angles are not right angles.

CIRCLES.

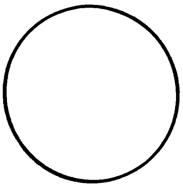
169. A *circle* is a plane figure bounded by a curved line, every point of which is equally distant from a point within, called the center.

170. The curve which bounds the circle is called the *circumference.* Any portion of the circumference is called the *arc.*

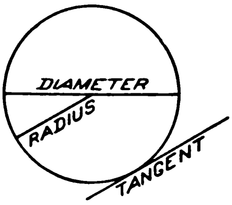
171. The *diameter* of a circle is a straight line drawn through the center and terminating in the circumference. A *radius* is a straight line joining the center with the circumference;

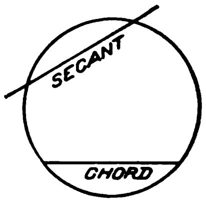
it has a length equal to one-half the diameter. All radii (plural of radius) are equal since every point of the circumference is equally distant from the center. All diameters are equal because a diameter is equal to two radii.

172. From the above it follows that two circles are equal when they have equal radii; if one circle is placed on the other so that the centers are in the same position, the circumferences will coincide.



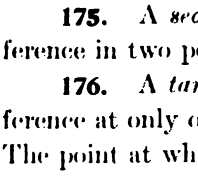
CIRCLE.

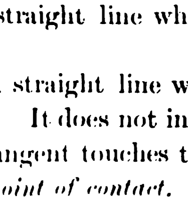




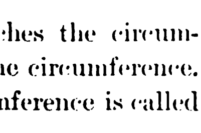
173. An arc equal to one-half the circumference is called a *semi-circumference*. A *quadrant* is one-quarter of a circumference. A quadrant may mean the sector, arc or angle.

174. A *chord* is a straight line joining the extremities of an arc. It is a line drawn across a circle that does not pass through the center.





CONCENTRIC CIRCLES.



INSCRIBED POLYGON.

175. A *secant* is a straight line which intersects the circumference in two points.

176. A *tangent* is a straight line which touches the circumference at only one point. It does not intersect the circumference. The point at which the tangent touches the circumference is called the *point of tangency* or *point of contact*.

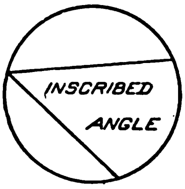
It may be proved that a tangent is perpendicular to the extremity of a radius.

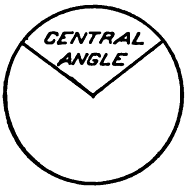
177. A *sector* of a circle is the portion or area included between an arc and two radii drawn to the extremities of the arc.

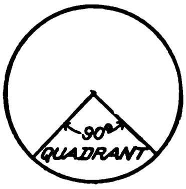
178. A *segment* of a circle is the area included between an arc and its chord.

179. Circles are *tangent* when the circumferences touch at only one point and are *concentric* when they have the same center.

180. An *inscribed polygon* is one whose vertices lie in the circumference and whose sides are chords.







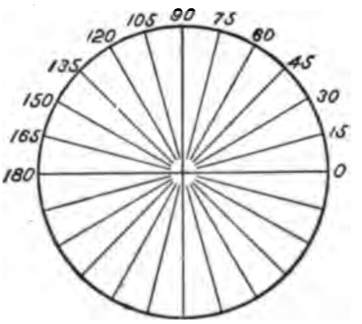
181. An *inscribed angle* is an angle whose vertex lies in the circumference and whose sides are chords. It is measured by one-half the intercepted arc.

182. A *central angle* is an angle whose vertex is at the center of the circle and whose sides are radii.

MEASUREMENTS.

ANGLES.

183. To measure an angle, describe an arc of any convenient radius having the center at the vertex of the angle. The arc included between the sides is the measure of the angle. With a given radius, the greater the divergence, the longer the arc. If several arcs are drawn having different radii, the intercepted arcs will have *different lengths* but they will all be the *same fraction* of the entire circumference.



184. To state the size of an angle as a fraction of the circumference is not always convenient, hence the entire circumference is divided into 360 equal parts called **degrees.** Angles are then measured in degrees. If an angle is measured by $\frac{1}{6}$ of the circumference it contains 60 degrees, because $\frac{360}{6} = 60$. An angle of 45 degrees is measured by $\frac{1}{8}$ of the circumference.

185. For accurate measurement each degree is divided into 60 equal parts called **minutes** and each minute into 60 equal parts called **seconds.**

The symbols of degrees, minutes and seconds are °, ′, ″, respectively. To express 23 degrees, 47 minutes and 9 seconds, we write 23° 47′ 9″.

186. According to Art. 139, if lines are perpendicular to each other, right angles are formed. A right angle is measured by $\frac{1}{4}$ the circumference, hence it is measured by an arc of 90° since $\frac{360}{4} = 90$.

187. Two angles are **complementary** when their sum is equal to one right angle or 90 degrees; they are **supplementary** when their sum is 180 degrees. Thus, an angle of 33 degrees is the complement of one of 57 degrees because $33 + 57 = 90$ or $90 - 57 = 33$.

Angles of 148 degrees and 32 degrees are supplementary because $148 + 32 = 180$.

EXAMPLES FOR PRACTICE.

1. How many seconds in 180 degrees? Ans. 648,000.
2. How many right angles in 202° 30′? Ans. $2\frac{1}{4}$.
3. What is the complement of 37°?
4. What is the complement of 11° 47′ 3″? Ans. 78° 12′ 57″.
5. What is the supplement of 131° 4′ 27″? Ans. 48° 55′ 33″.
6. One angle of a scalene triangle is 17°; another angle is 21°. What number of degrees is the third? Ans. 142°.
7. The vertical angle between the equal sides of an isosceles triangle is 25 degrees. How many degrees in each of the base angles? Ans. 77° 30′.

AREAS OF POLYGONS.

RECTANGLES.

188. The amount of space covered by a plane figure is called its **area.** In order to express area, we state how many times greater it is than another area taken as a unit.

189. The *unit* most commonly used is the **square foot,** or an area equal to a square measuring one foot on a side. For small areas the **square inch** may be used. If we say that a certain lot of land contains 10,000 square feet, we mean that its area is 10,000 times as great as an area of one square foot.

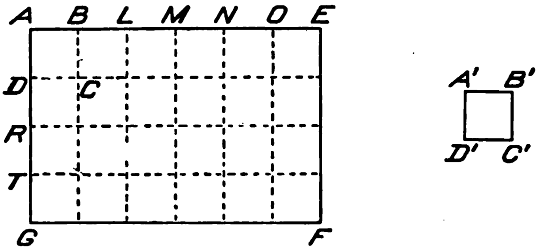


Fig. 2.

190. In Fig. 2 let A′ B′ equal 1 inch and A′ D′ equal 1 inch; then the area A′ B′ C′ D′ equals 1 square inch. The divisions A B, B L, L M, M N, N O and O E are each equal to A′ B′. Similarly the divisions A D, D R, R T and T G are equal to A′ D′.

It will be seen that the rectangle A E F G contains 24 of these small squares, each equal to the unit area A′ B′ C′ D′. Since there are 6 divisions in the base and 4 in the altitude, and 6 × 4 = 24, we have the following rule:

191. To find the area of a rectangle, multiply the base by the altitude.

Example. The base of a rectangle is 18 feet and the altitude is 7 feet; what is the area?

$$\textit{base} \times \textit{altitude} = \textit{area}$$

$$18 \times 7 = 126$$

Ans. 126 square feet.

Another Example. The *area* of a rectangle is 39 square inches and the base is 13 inches; what is the altitude?

$$\textit{base} \times \textit{altitude} = \textit{area}$$

or

$$\textit{altitude} = \frac{\textit{area}}{\textit{base}}$$

$$= \tfrac{39}{13} = 3$$

Ans. 3 inches.

192. Since the *square* is a rectangle, its area is found by the same rule. In the square, however, the base equals the altitude; hence the area is equal to the square of one side.

PARALLELOGRAMS.

193. In the parallelogram shown in Fig. 3 we know that the base is D C, and the altitude A G, or any line perpendicular to D C. If we produce D C to F to meet the line perpendicular to the bases from E, the right triangle E F C is formed. This triangle may be proved equal to the right triangle A G D. If we take the area A G D from the parallelogram and add the equal area E F C, we do not change the area of the parallelogram. Hence,

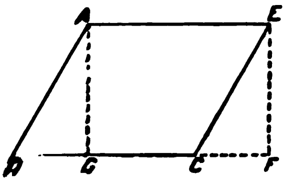


Fig. 8.

area A E F G = area A E C D.

We know that the area of the rectangle A E F G is equal to the product of its base and altitude. From this we can state:

194. The area of a parallelogram is equal to the product of the base and altitude.

Example. The altitude of a parallelogram is 9 inches and the base is 12 inches. What is the area?

area = base × altitude
area = 12 × 9
area = 108 square inches.

195. $\text{Base} = \dfrac{\text{Area}}{\text{Altitude}}$ $\text{Altitude} = \dfrac{\text{Area}}{\text{Base}}$.

TRIANGLES.

196. If we draw the diagonal A F (Fig. 4), the parallelogram A B F C is divided into two equal triangles — B A F and A F C. The area of the parallelogram is equal to the base multiplied by the altitude. Hence the area of the triangle is equal to ½ the base multiplied by the altitude. This may be expressed

$$\text{area} = \frac{\text{base} \times \text{altitude}}{2}$$
$$= \frac{\text{base}}{2} \times \text{altitude}$$
$$= \text{base} \times \frac{\text{altitude}}{2}$$

197. The area of a triangle equals one-half the product of the base and the altitude.

Example. The base of a triangle is 6 and the altitude is 3. What is the area?

$$\text{area} = \frac{\text{base} \times \text{altitude}}{2}$$

$$= \frac{6 \times 3}{2} = 9.$$

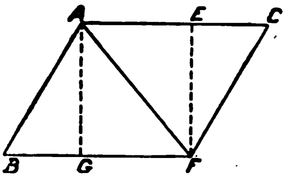


Fig. 4.

198. In case the **three sides are known** the area is found as follows:

(1) Add together the three sides and divide the sum by 2.

(2) Subtract from the half sum (1) the three sides severally.

(3) Multiply the half sum (1) and the three remainders (2), and extract the square root of the product.

Example. The three sides of a triangle are 24, 18 and 16 inches respectively. What is the area?

$$\text{The half sum (1) is} \quad \frac{24 + 18 + 16}{2} = 29$$

$$\text{The three remainders (2) are} \begin{cases} 29 - 24 & = 5 \\ 29 - 18 & = 11 \\ 29 - 16 & = 13 \end{cases}$$

$$\begin{aligned} \text{The area} &= \sqrt{29 \times 5 \times 11 \times 13} \\ &= \sqrt{20{,}735} \\ &= 144 \text{ — square inches.} \end{aligned}$$

199. The Right Triangle. The area of the right triangle is equal to one-half the product of the base and altitude. If both are given, the problem is very simple; if the base and hypothenuse are given, the altitude may be found by the principle explained in Arts. 158 and 159.

$$\text{area} = \frac{\text{A B} \times \text{A C}}{2}. \quad \text{(See Fig. 5.)}$$

200. The Isosceles Triangle. The isosceles triangle A B C is equal to the sum of the two right triangles A B D and B C D.

If the base and altitude are known, the principle explained in Art. 197 may be used.

If the base A C and the side A B are known, the principle explained in Art. 159 may be followed to find the altitude, and then the area is found according to Art. 197. As the side B C = side A B, the area may be found according to Art. 198, as the three sides are given.

201. The Equilateral Triangle. The equilateral triangle is an isosceles triangle; hence any of the above rules may be used according to the data.

Suppose the side B C = 1. The side A C also equals 1, and D C = $\frac{1}{2}$.

$$\begin{aligned}\text{From Art. 159, } \overline{BD}^2 &= \overline{BC}^2 - \overline{DC}^2\\ &= 1^2 - (\tfrac{1}{2})^2\\ &= 1 - \tfrac{1}{4} = \tfrac{4}{4} - \tfrac{1}{4}\\ &= \tfrac{3}{4}\\ BD &= \sqrt{\tfrac{3}{4}} = \sqrt{.75}\\ &= .866\end{aligned}$$

202. To find the altitude multiply the side by .866.

$$\begin{aligned}\text{Area of triangle } ABC &= \frac{BD}{2} \times AC\\ &= (.433 \times AC) \times AC\\ &= .433 \times \overline{AC}^2\end{aligned}$$

203. The area of an equilateral triangle is equal to the square of the side multiplied by .433.

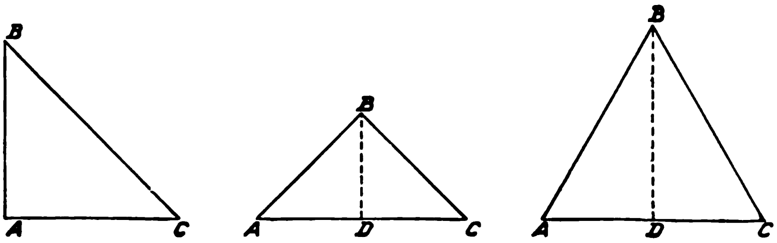


Fig. 5.

The above rule will be found very useful, on account of its simplicity, and because the results are sufficiently accurate for all practical purposes.

Example. The side of an equilateral triangle is 6 inches. What is the area?

$$\begin{aligned} \text{Area} &= (\text{side})^2 \times .433 \\ &= 6^2 \times .433 \\ &= 36 \times .433 \\ &= 15.588 \text{ square inches.} \end{aligned}$$

Another example. The side of an equilateral triangle is 8 inches. What is the area?

$$\text{B C} = 8 \qquad\qquad \text{D C} = 4$$

$$\begin{aligned} \text{B D} &= \sqrt{64 - 16} \\ &= \sqrt{48} \\ &= 6.928 \\ \text{area} &= \frac{\text{B D} \times \text{A C}}{2} \\ &= \frac{6.928 \times 8}{2} \\ &= 27.71 \text{ square inches.} \end{aligned}$$

204. $\text{Base} = \dfrac{2 \times \text{Area}}{\text{Altitude}} \qquad \text{Altitude} = \dfrac{2 \times \text{Area}}{\text{Base}}.$

EXAMPLES FOR PRACTICE.

1. The base of a right-angled triangle is 12 feet and the hypothenuse is 20 feet. What is the length of the perpendicular? Ans. 16 feet.

2. The base of a parallelogram is 16.25 feet and the altitude is 7.5 feet. What is the area? Ans. 121.875 square feet.

3. The foot of a ladder is 15 feet from the base of a building, and the top reaches a window 36 feet above the base. What is the length of the ladder? Ans. 39 feet.

4. An equilateral triangle has an area of 21.217 square inches. What is the length of the side? Ans. 7 inches.

5. An isosceles triangle has a base 16 inches long and the equal sides 18 inches long. What is the area? Ans. 128.96 square inches.

6. The base of a triangle is 20 feet and the altitude is 18 feet. What is the side of a square having the same area? Ans. 13.41 feet.

7. What is the area of a triangle having sides **16, 12** and 10 feet? **Ans. 59.9 square feet.**

8. What is the area of an equilateral triangle if **the side is** 11 feet? Ans. **52.393 square feet.**

TRAPEZOID.

205. If in the trapezoid shown in Fig. 6, a diagonal **D B is** drawn, the trapezoid is divided into **two** triangles. The altitude of both **triangles** is the same; the base of triangle **D B C** is the lower base and the base of **D A B is** the upper base. To **find** the **area we** proceed as follows:

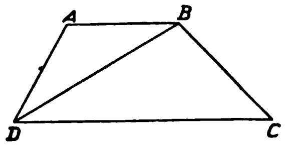


Fig. 6.

$$\text{Area D B C} = \text{altitude} \times \frac{1}{2}\text{D C}.$$

$$\text{Area D A B} = \text{altitude} \times \frac{1}{2}\text{A B}.$$

$$\text{Total area} = \text{altitude} \times \frac{\text{D C} + \text{A B}}{2}.$$

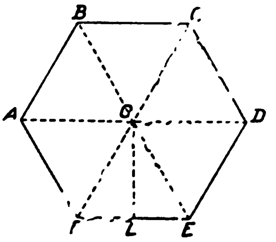
206. The area of a trapezoid equals the product of the altitude and one-half the sum of the bases.

Example. If the bases of a trapezoid are 10 inches and **8** inches respectively and the altitude is 7 inches, what is the area?

$$\text{Area} = 7\left(\frac{10+8}{2}\right) = 63 \text{ square inches.}$$

HEXAGON.

207. A regular polygon is one all of whose sides and are equal. We may divide a regular hexagon into six equal equilateral triangles. The area of the triangle F O E may be found by any of the rules given for equilateral triangles.



$$\text{O L} = \sqrt{\overline{\text{O E}}^2 - \overline{\text{L E}}^2}$$

$$\text{Area} = \frac{\text{O L} \times \text{F E}}{2}$$

$$\text{Area of hexagon} = \frac{\text{O L} \times \text{F E}}{2} \times 6 = \text{O L} \times \text{F E} \times 3$$

We know that the area of an equilateral triangle may be found by multiplying the square of the side by .433.

$$\text{Area} = (\text{side})^2 \times .433.$$

$$\text{Area of hexagon} = (\text{side})^2 \times .433 \times 6.$$

$$= (\text{side})^2 \times 2.598.$$

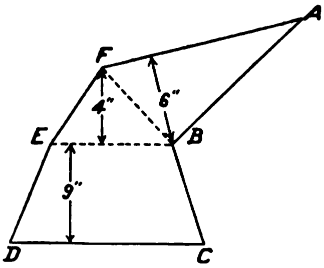
Example. The side of regular hexagon is 5 feet long. **What** is the area?

$$5^2 \times 2.598 = 64.95 \text{ square feet.}$$

POLYGONS.

208. To find the area of a polygon we divide it into triangles and take the sum of the areas.

209. Suppose we wish to know the area of the given polygon A B C D E F. First we divide it into the triangles A B F and E F B and the *trapezoid B C D E. This is done by drawing the lines F B and E B. We find the altitudes to be 6 inches, 4 inches and 9 inches respectively. If the line A F is 14 inches, D C 15 inches and E B 11 inches, the areas are as follows:



$$\text{Area A B F} = \frac{6 \times 14}{2} = 42 \text{ square inches.}$$

$$\text{Area B E F} = \frac{4 \times 11}{2} = 22 \text{ square inches.}$$

$$\text{Area B C D E} = \frac{9 \times (11 + 15)}{2} = 117 \text{ square inches.}$$

The total area is 42 + 22 + 117, or 181 square inches.

EXAMPLES FOR PRACTICE.

1. The bases of a trapezoid are 23 feet and 11 feet; the altitude is 9 feet. What is the area?

Ans. 153 square feet.

2. What is the area of a regular hexagon having a perimeter of 42 feet?

Ans. 127.3 square feet.

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* In this particular case E B happens to be parallel to D C and we accordingly find the area of B C D E in accordance with Art. 205.

3. *Find the area of a trapezium, a diagonal of which measures 42 feet, and the perpendicular from one vertex to the diagonal is 16 feet, and from the other vertex 18 feet.

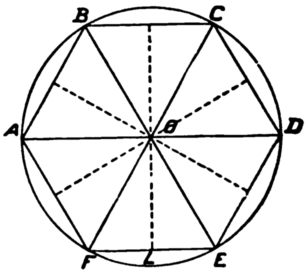
Ans. 714 square feet.

4. What is the area of the trapezium a diagonal of which is 35 feet 6 inches long, the perpendiculars to the diagonal from the other two vertices being 9 and 3 feet respectively?

Ans. 213 square feet.

CIRCLE.

210. Suppose we have a regular hexagon A B C D E F inscribed in the circle whose center is O. The hexagon may be divided into six triangles by drawing the radii connecting the vertices A, B, C, D, etc., with the center. These triangles are all equal and the area of each is equal to one-half the product of the base and altitude. We may say that it is equal to one-half the altitude multiplied by the base. The area of six triangles is equal to one-half the altitude O L of one of them multiplied by six times the base or the perimeter.



$$\text{Area} = \frac{\text{altitude}}{2} \times \text{perimeter}.$$

211. Now if the number of sides of the polygon is increased, the altitude will increase and the perimeter will be more nearly equal to the circumference of the circle. If the number of sides is increased indefinitely, the altitude of each triangle will be practically the same as the radius of the circle and the perimeter will be very nearly the same as the circumference. The area of the circle will be that of the inscribed polygon of an infinite number of sides, or

$$\text{Area of circle} = \frac{\text{radius}}{2} \times \text{circumference}.$$

It has been found that the diameter of a circle multiplied by the constant 3.1416 is equal to the circumference. The constant 3.1416 expresses the ratio of the circumference to the diameter, and is called Pi; it is usually designated by the Greek letter π.

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*Note. The diagonal connects two vertices. The perpendiculars are from the other two vertices to the diagonal. They are the altitudes of the two triangles of which the diagonal forms the base.

212. The circumference of a circle *is equal to the diameter multiplied by* 3.1416.

From the above we know that

$$\begin{aligned}\text{Area of circle} &= \frac{\text{radius}}{2} \times \text{circumference.}\\ &= \frac{\text{radius}}{2} \times \text{diameter} \times 3.1416.\\ &= \frac{\text{radius}}{2} \times 2 \times \text{radius} \times 3.1416.\\ &= \text{radius} \times \text{radius} \times 3.1416.\end{aligned}$$

213. The area of a circle *equals the square of the radius multiplied by* 3.1416.

Example. What is the area of a circle whose radius is 4 feet?

$$\begin{aligned}\text{Area} &= (\text{radius})^2 \times 3.1416\\ &= 4 \times 4 \times 3.1416\\ &= 50.2656 \text{ square feet.}\end{aligned}$$

Another example. If the diameter of a circle is 12 inches, what is the area?

$$\begin{aligned}\text{Radius} &= \frac{\text{diameter}}{2}\\ &= \frac{12}{2} = 6 \text{ inches.}\\ \text{Area} &= 6 \times 6 \times 3.1416\\ &= 113.0976 \text{ square inches.}\end{aligned}$$

The above example may also be solved as follows:

$$\begin{aligned}\text{Area} &= \text{radius} \times \text{radius} \times 3.1416\\ &= \frac{\text{diameter}}{2} \times \frac{\text{diameter}}{2} \times 3.1416\\ &= \frac{\text{diameter} \times \text{diameter}}{4} \times 3.1416\\ &= \text{diameter} \times \text{diameter} \times \frac{3.1416}{4}\\ &\text{Since } \frac{3.1416}{4} = .7854.\\ \text{Area} &= 12 \times 12 \times .7854\\ &= 113.0976 \text{ square inches.}\end{aligned}$$

153.; k

214. The area of a circle *is equal to the square of the diameter multiplied by* .7854.

215. By considering Art. 211 to Art. 214 **and using the** principles of Elementary Algebra we have the **following:**

$$\textbf{Circumference} = 3.1416 \times \textit{diameter}$$
$$= 2 \times 3.1416 \times \textit{radius}$$
$$= \frac{4 \times \textit{area}}{\textit{diameter}}$$

$$\textbf{Area} = \textit{radius} \times \textit{radius} \times 3.1416$$
$$= \textit{diameter} \times \textit{diameter} \times .7854$$
$$= \textit{circumference} \times \frac{\textit{radius}}{2}$$

$$\textbf{Diameter} = \frac{\textit{circumference}}{3.1416}$$
$$= \sqrt{\frac{4 \times \textit{area}}{3.1416}}$$

$$\textbf{Radius} = \frac{\textit{circumference}}{2 \times 3.1416}$$
$$= \sqrt{\frac{\textit{area}}{3.1416}}$$

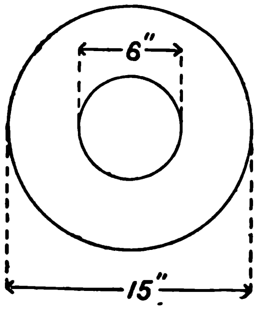


Fig. 7.

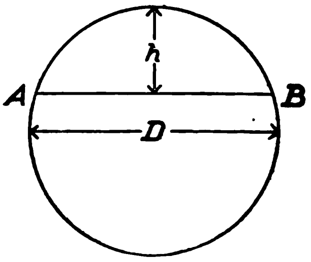


Fig. 8.

216. Circular Ring. To obtain the area of a flat circular ring, first find the area of the hole and subtract it from the area enclosed by the larger circumference.

In Fig. 7 we have:

Area of large circle $= .7854 \times 15^2 =$ **176.71 square inches**

Area of small circle $= .7854 \times 6^2 =$ 28.27 square inches

Area of ring = **148.44** square inches

217. Sector. In finding **the area of a sector we determine**

the fraction of the area of a circle. Thus if the sector has an arc of 60°, the area of the sector would be 60 ÷ 360 = one-sixth of the area of the circle of which the sector forms a part. Similarly a sector having an arc of 45° has one-eighth the area of a circle having a radius equal to one of the sides of the sector.

218. Segment. There are many rules and tables for finding the area of a segment. The following formula is very simple and gives approximate results. The derivation of this formula is very difficult, as it involves higher mathematics; the formula may be used, however, by students and engineers who wish a simple formula giving approximate results.

$$\text{Area of segment A} = \frac{4\,h^2}{3}\sqrt{\frac{D}{h} - .608}$$

In this formula h = height of segment and D = the diameter of the circle. See Fig. 8.

219. Suppose we wish to find the area of a segment when the diameter of the circle is 12 feet and the height of the segment is 4 feet.

In this case h = 4 feet and D = 12 feet. Substituting these values in the formula we have,

$$\begin{aligned} A &= \frac{4\,h^2}{3}\sqrt{\frac{D}{h} - .608} \\ &= \frac{4 \times 4^2}{3}\sqrt{\frac{12}{4} - .608} \\ &= \frac{64}{3}\sqrt{3 - .608} \\ &= \frac{64}{3}\sqrt{2.392} = \frac{64}{3} \times 1.55 \\ &= 33.06 + \text{square feet.} \end{aligned}$$

EXAMPLES FOR PRACTICE.

1. The diameter of a circle is 9 feet; what is the area?

Ans. 63.617 square feet.

2. What is the circumference of a circle 17 inches in diameter?

Ans. 53.407+ inches.

3. If a circle is 4 feet in diameter, what is the area of a sector having an arc of 80°? Ans. 2.7925 square feet.

4. The area of a certain circle is **283.53 square feet**; **what** is the circumference?

Ans. 59.69 feet.

5. The area of a circle is **99.402 square inches; what is the** circumference?

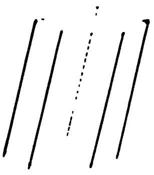
Ans. 35.343 inches.

SOLIDS.

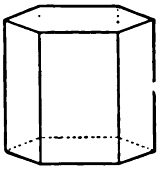
220. A solid has three dimensions—**length, breadth** and thickness.

PRISMS.

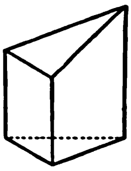
221. A prism is a solid having **two opposite faces called** *bases*, equal and parallel; the other faces, called **lateral faces, are** parallelograms.



PRISM.



RIGHT PRISM.



TRUNCATED PRISM.

222. The *altitude* of a prism is the perpendicular **distance** between the bases.

223. The combined area of all the lateral faces is called **the** *lateral area*.

The *total area* or entire surface is the lateral area plus **the** area of the bases.

224. The intersections of the faces are called the *lateral edges*.

225. Prisms are *triangular*, *rectangular*, *hexagonal*, etc., according to the shape of the bases.

226. A *right prism* is one whose lateral edges are perpendicular to the bases.

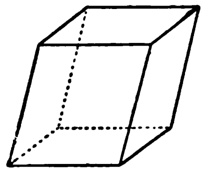
227. A *regular prism* is a right prism having regular polygons for bases.

228. A *parallelopiped* is a prism whose **bases are parallelograms**. If all the edges are perpendicular to the **bases, it is** called a right parallelopiped.

229. A *rectangular* parallelopiped is a right parallelopiped whose bases are rectangles; all the faces are rectangles.

230. A *cube* is a rectangular parallelopiped all of whose faces are squares.

231. A *truncated* prism is the portion of a prism included between its base and a plane cutting all the lateral edges.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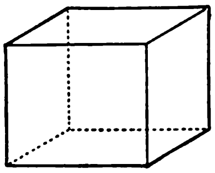
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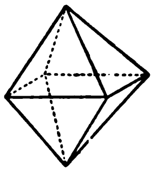
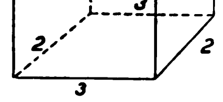

OCTAHEDRON.

232. Lateral Area. Suppose we have a box 3 feet long, 2 feet wide and 6 feet deep. Let us consider the top and bottom as the bases.



The area of the front face is evidently $6 \times 3 = 18$ square feet; the area of the back face is also 18 square feet, as it is equal to the front face.

The area of the left-hand face is $2 \times 6 = 12$ square feet, and the area of the right-hand face is the same.

The convex surface is equal to

$$\begin{aligned} 2 \times 18 &= 36 \\ 2 \times 12 &= 24 \\ \hline &\ 60 \text{ square feet.} \end{aligned}$$

233. It is readily seen that had we multiplied the perimeter of the base by the altitude we would have obtained the same result.

Perimeter of base = 3 + 2 + 3 + 2 = 10 feet.
Altitude = 6 feet.
Lateral Area = 10 × 6 = 60 square feet.

234. ***The lateral area of any right prism is equal to the product of the perimeter of the base and altitude.***

This rule may be used for all right prisms, whether the base is a triangle, square, parallelogram or hexagon.

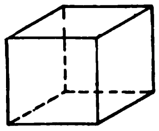
Example. What is the lateral area of a right prism whose altitude is 11 feet and each side of the hexagonal base is 4 feet?

The perimeter of the base $= 6 \times 4 = 24$ feet.
Altitude $= 11$ feet.
Lateral area $= 24 \times 11 = 264$ square feet.

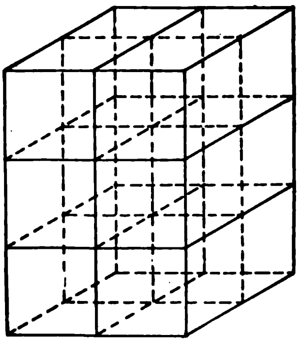
235. Total Area. *To find the total area of a prism, add the areas of the bases to the lateral area.*

236. In the prism described in Art. 232, the lateral area was found to be 60 square feet. The area of the lower base is equal to $3 \times 2 = 6$ square feet and the area of the upper base is the same.

Total area $= 60 + (2 \times 6) = 72$ square feet.



A



B

237. Volume. The unit of volume is a cube whose edges are each equal to the unit of length (inch, foot, etc.). The volume is expressed as being equal to the number of times the unit volume is contained in the prism.

238. By considering the above figures we see that the unit of volume A is contained 4 times in each layer of the prism B. As there are 3 layers, it is contained $4 \times 3 = 12$ times. If the unit of volume is the cubic foot, the prism B would contain 12 cubic feet. The area of the base is 4 square feet and the altitude is 3 feet. Hence we may say:

239. *The volume of any right prism is equal to the product of the area of the base and the altitude.*

Example. Find the volume of a block of stone 3 feet long, 2 feet wide and 7 feet high.

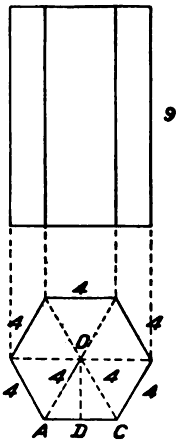
The area of the base $= 2 \times 3 = 6$ square feet.
The volume $= 6 \times 7 = 42$ cubic feet.

We may consider any other side as the base ; the result is the same.

$$2 \times 3 \times 7 = 42.$$
$$7 \times 3 \times 2 = 42.$$
$$7 \times 2 \times 3 = 42.$$

240. Let us consider an *hexagonal prism* having an altitude of 9 feet and each side of the base 4 feet.

To find the volume we must first find the area of the base. This may be done according to Art. 207, or by finding the altitude O D of the triangle A O C, and then proceeding as in Art. 197. After finding the area of the base, multiply by the altitude.



Area of base $= 16 \times 2.598$
$= 41.568$ square feet
Volume $= 41.568 \times 9$
$= 374.112$ cubic feet.

241. The lateral and total areas are found as before. (See Arts. 232 and 235.)

Lateral Area $= 24 \times 9$
$= 216$ square feet
Total Area $= 216 + (2 \times 41.568)$
$= 299.136$ square feet.

EXAMPLES FOR PRACTICE.

1. Find the volume of a cube whose edges are each 6 feet 6 inches.

Ans. $274\frac{5}{8}$ cubic feet.

2. Find the volume of a triangular prism whose altitude is 20 feet and each side of the base 4 feet.

Ans. 138.56+ cubic feet.

3. Find the lateral area of a prism having an altitude of 7 feet and its base a pentagon, each side of which is 4 feet.

Ans. 140 square feet.

4. What is the entire surface of a parallelopiped 8 feet 9 inches long, 4 feet 8 inches wide and 3 feet 3 inches high?

Ans. $168\frac{7}{8}$ square feet.

5. Find the volume of an hexagonal prism 17 feet high, having a base each side of which is 7 feet.

Ans. 2,164.1 cubic feet.

6. What is the total area of a cube whose edges are each 7 inches?

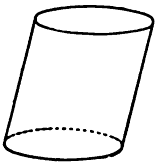
Ans. 294 square inches.

7. What is the weight of a cast iron block 6 inches long, 5 inches wide and 3 inches high? Cast iron weighs about .26 pound per cubic inch.

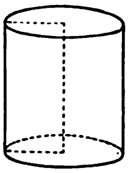
Ans. 23.4 pounds.

CYLINDERS.

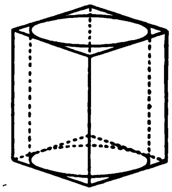
242. A *cylinder* is a solid bounded by a cylindrical surface and two parallel planes intersecting this surface. It may be considered as being a prism whose base is a polygon of an infinite number of sides.



CYLINDER.



RIGHT CYLINDER.



INSCRIBED CYLINDER.

243. The parallel faces are called the *bases.*

244. The *altitude* is the perpendicular distance between the bases.

245. A *circular cylinder* is a cylinder whose base is a circle.

246. A *right circular cylinder,* or cylinder of revolution, is a cylinder generated by the revolution of a rectangle about one side as an axis.

247. A cylinder may be inscribed in or circumscribed about a prism.

248. The cylindrical area is called the *lateral area.* The *total area* is the sum of the lateral area and the areas of the bases.

249. Lateral Area. *The lateral area of a cylinder is equal to the circumference of the base multiplied by the altitude.*

250. Suppose we have a cylinder 15 inches high and 6 inches diameter.

The circumference $= 3.1416 \times 6 = 18.8496$ inches.
The lateral area $= 18.8496 \times 15 = 282.744$ square inches.

251. Total Area. *To find the total area of a cylinder, add areas of the bases to the lateral area.*

252. In the above cylinder the area of the base is,

$36 \times .7854 = 28.2744$ square inches.
The total area $= 282.744 + 2\ (28.274)$
$= 339.292$ square inches.

253. Volume. *The volume of a cylinder is equal to the area the base multiplied by the altitude.*

Example. A cylinder is 6 feet in height and 4 feet in meter; what is the volume?

Area of base $= 4 \times 4 \times .7854$
$= 12.5664$ square feet.
Volume $= 12.5664 \times 6$
$= 75.3984$ cubic feet.

EXAMPLES FOR PRACTICE.

1. What is the volume of a cylinder whose altitude is 8$\frac{1}{2}$ t and base 3 feet in diameter?

Ans. 60.08+ cubic feet.

2. Find the total area of a cylindrical boiler 56 inches in meter and 16 feet long.

Ans. 268.78 square feet.

3. How many gallons of water will the above boiler hold if contains no tubes?

Ans. 2,047 gallons.

PYRAMID.

254. A pyramid is a solid whose base is a polygon and ose sides are triangles.

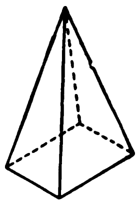
255. The vertices of the triangles meet to form the *vertex* the pyramid.

256. The *altitude* of the pyramid is the perpendicular dis- ce from the vertex to the base.

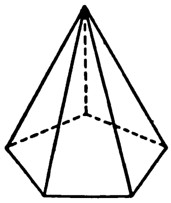
257. A pyramid is triangular, quadrangular, etc., according the shape of the base.

258. A *regular* pyramid is one whose base is a regular polygon and whose vertex lies in a perpendicular erected at the center of the base.

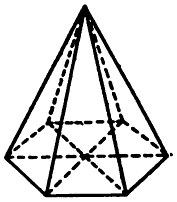
259. The *slant height* of a regular pyramid is a line drawn from the vertex perpendicular to a side of the base. It is the altitude of one of the triangles which form the sides.



PYRAMID.



REGULAR PYRAMID.



HEXAGONAL PYRAMID.

260. The lateral edges of a pyramid are the intersections of the faces.

261. Lateral Area. The lateral area is the combined area of all the triangles forming the sides.

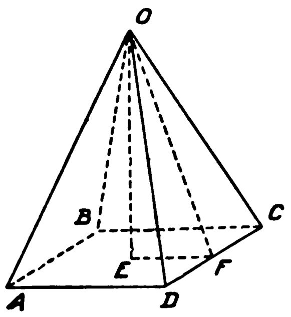


Fig. 9.

262. As the area of each triangle is equal to the product of the base and one-half the altitude, *the lateral area of the pyramid is equal to the perimeter multiplied by one-half the slant height.*

Example. Suppose a pyramid having a slant height of 14 feet has a regular pentagon for a base, each side of which is 5 feet. What is the lateral area?

Perimeter of base = 5 × 5 = 25 feet.
One-half the slant height = 7 feet.
Lateral area = 25 × 7 = 175 square feet.

263. If the slant height is not given it can usually be found by means of the principle explained in Art. 158.

264. The pyramid shown in Fig. 9 has an altitude O E of 12 feet and each side of the square base is 8 feet. We know that

EF = 4 feet. Since the angle O E F is a right angle (Art. 159),

$$\overline{OF}^2 = \overline{OE}^2 + \overline{EF}^2$$
$$= 144 + 16$$
$$= 160$$
$$OF = \sqrt{160} = 12.65 \text{ (nearly)}$$

$$\text{The lateral area} = \frac{12.65 \times 32}{2}$$
$$= 202.4 \text{ square feet.}$$

265. The Total Area *is equal to the lateral area plus the area of the base.*

In the above example the area of the base is 64 square feet. (Art. 192.) The total area is,

64 + 202.4 = 266.4 square feet.

266. Volume. *The volume of a pyramid is equal to the area of the base multiplied by one-third the altitude.*

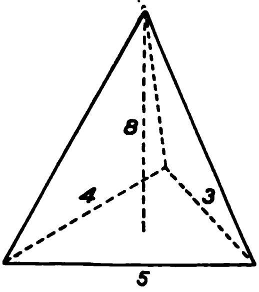


Fig. 10.

267. Example. Suppose the triangular pyramid shown in Fig. 10 has the following dimensions: altitude 8 feet, sides of base 5, 4 and 3 feet. What is the volume?

Area of base:

$$\text{The half sum (1)} = 12 \div 2 = 6$$

$$\text{The three remainders (2) are} \begin{cases} 6 - 5 = 1 \\ 6 - 4 = 2 \\ 6 - 3 = 3 \end{cases}$$

$$\text{The area} = \sqrt{6 \times 1 \times 2 \times 3}$$
$$= \sqrt{36}$$
$$= 6 \text{ square feet.}$$

$$\text{Volume} = \text{area of base} \times \tfrac{1}{3} \text{ altitude}$$
$$= 6 \times \tfrac{8}{3}$$
$$= 16 \text{ cubic feet.}$$

EXAMPLES FOR PRACTICE.

1. A triangular pyramid is 9 inches in height and each side of the base is 4 inches long. Find the volume.

Ans. 20.784 cubic inches.

2. Find the lateral area, total area and volume of a pyramid having a slant height of 72 inches and a square for a base, the side of which is 8 feet.

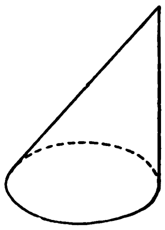
$$\text{Ans.}\begin{cases}\text{Lateral Area} = 96 \text{ square feet.}\\ \text{Total Area} = 160 \text{ square feet.}\\ \text{Volume} = \dfrac{64\sqrt{20}}{3} = 95.4 \text{ cubic feet.}\end{cases}$$

3. Find the lateral area and volume of a hexagonal pyramid each side of whose base is 6 feet and whose altitude is 12 feet.

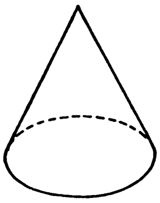
$$\text{Ans.}\begin{cases}\text{Lateral Area} = 235.26 \text{ square feet.}\\ \text{Volume} = 374.112 \text{ cubic feet.}\end{cases}$$

CONES.

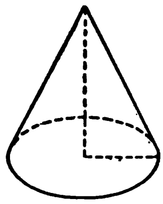
268. A cone is a solid bounded by a conical surface and a plane which cuts the conical surface. It may be considered as a pyramid of an infinite number of sides.



CONE.



CIRCULAR CONE.



CONE OF REVOLUTION.

269. The curved surface is called the *lateral area*, and the plane is called the *base*. The convex surface tapers to a point called the *vertex*.

270. The *altitude* of a cone is the perpendicular distance from the vertex to the base.

271. An *element* of a cone is any straight line from the vertex to the perimeter of the base.

272. A *circular cone* is a cone whose base is a circle.

273. A *right circular cone*, or *cone of revolution*, is a cone whose axis is perpendicular to the base. It may be generated by the revolution of a right triangle about one of the sides as an axis.

274. Lateral Area. The lateral area of a cone is found in the same way as in the case of a pyramid. Multiply the perimeter of the base by one-half the slant height.

Example. The base of a cone is 12 inches in diameter and the slant height is 16 inches. What is the lateral area?

$$\begin{aligned}\text{Perimeter of base} &= 3.1416 \times 12\\ &= 37.699 \text{ inches}\\ \text{Lateral area} &= 37.699 \times \frac{16}{2}\\ &= 301.592 \text{ square inches.}\end{aligned}$$

275. To find the total area, add the area of the base.

In the above example, the area of the base is,

$$3.1416 \times 36 = 113.0976$$

$$\text{Total area} = 113.0976 + 301.592 = 414.6896 \text{ square inches.}$$

276. Volume. The volume of a cone is equal to the product of the area of the base and one-third the altitude.

Example. The altitude of a cone is 18 inches and the radius of the base is 2 inches. What is the volume?

$$\begin{aligned}\text{Area of base} &= 3.1416 \times 4 = 12.5664\\ \text{Volume} &= 12.5664 \times 6\\ &= 75.3984 \text{ cubic inches.}\end{aligned}$$

277. In case the altitude is not known, it may be found if the slant height and radius of base are known.

EXAMPLES FOR PRACTICE.

1. Find the total area of a cone having an altitude of 21 feet and a base 16 feet in diameter.

Ans. 765.75 square feet.

2. Find the volume of a cone if the area of the base is 12.566 square inches, and the slant height is 6¼ inches.

Ans. 25.898+ cubic inches.

3. The volume of a cone is 1,061.84 cubic inches and the altitude is 24 inches. What is the lateral area?

Ans. 507.64+ square inches.

FRUSTUMS OF PYRAMIDS.

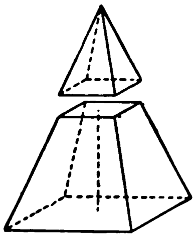
278. The frustum of a pyramid is the solid formed by passing a plane parallel to the base and cutting all the edges.

279. The upper plane of the frustum is called the *upper base* and the lower plane the *lower base.*

280. The *altitude* is the perpendicular distance between the bases.

281. To find the **lateral area** of the frustum, multiply one-half the sum of the two perimeters by the slant height.

282. Example. The lower base of a frustum of a square pyramid measures 12 inches on each side and the upper base measures 4 inches on a side; if the slant height is 7 inches, what is the lateral area?



FRUSTUM OF PYRAMID.

Perimeter of lower base = 4 × 12 = 48 inches

Perimeter of upper base = 4 × 4 = 16 inches

One-half the sum = $\frac{48 + 16}{2}$ = 32 inches

The lateral area = 7 × 32 = 224 square inches.

283. The total area is found by adding the areas of the bases to the lateral area.

In the above example we have:

Area of lower base = 144 square inches
Area of upper base = 16 square inches
Lateral area = 224 square inches
Total area = 384 square inches.

284. Volume. To find the volume of a frustum of a pyramid: (1) Find the areas of the two bases and add them together. (2) To this quantity (1) add the *square root* of the *product* of the two bases. (3) Multiply the sum thus obtained by one-third the altitude.

285. Example. The altitude of the frustum of a square pyramid is 10 feet. Each side of the lower base is 12 feet and each side of the upper base is 9 feet. What is the volume?

Area of lower base = 12 × 12 = 144 square feet
Area of upper base = 9 × 9 = 81 square feet
Sum (1) = 225 square feet

Square root of product (2) = $\sqrt{144 \times 81}$

Volume = $\frac{10}{3} \times (225 + 108)$
= 1110 cubic feet.

FRUSTUMS OF CONES.

286. The **frustum** of a cone is a solid formed by a plane parallel to the base and cutting the conical surface.

287. The upper plane is called the *upper base* and the lower plane the *lower base*.

288. The *altitude* is the perpendicular distance between the bases.

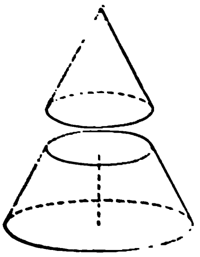
289. To find the **lateral area** of the frustum, multiply one-half the sum of the two perimeters by the slant height.

290. Example. The lower base of a frustum of a cone is 11 inches in diameter and the upper base is 8 inches in diameter. What is the convex area if the slant height is 5 inches?

Circumference of lower base $= \pi d$ (Art. 215)
$= 3.1416 \times 11$
$= 34.5576$ inches
Circumference of upper base $= 3.1416 \times 8$
$= 25.1328$ inches

$$\frac{34.5576 + 25.1328}{2} = 29.8452$$

Lateral area $= 29.8452 \times 5$
$= 149.2260$ square inches.



FRUSTUM OF CONE.

291. The **total area** is equal to the areas of the bases added to the lateral area. Thus in the above example,

Area of upper base $= \frac{\pi}{4} d^2$ (Art. 215)
$= .7854 \times 64$
$= 50.2656$ square inches
Area of lower base $= .7854 \times 121$
$= 95.0334$ square inches

$$\begin{array}{r} 50.2656 \\ 95.0334 \\ 149.2260 \\ \hline 294.5250 \end{array}$$

Total area $= 294.5250$ square inches.

292. The **volume** of the frustum is found in the same manner as the frustum of a pyramid.

(1.) Find the areas of the two bases and add them together. (2.) To this quantity (1) add the *square root* of the *product* of the two bases. (3.) Multiply the sum thus obtained by one-third the altitude.

293. Example. The altitude of the frustum of a cone is 18 inches. The upper base is 3 inches in radius and the lower base is 12 inches in diameter. What is the volume?

$$\begin{aligned}
\text{Area of lower base} &= 3.1416 \times 36 \\
&= 113.0976 \\
\text{Area of upper base} &= 3.1416 \times 9 \\
&= 28.2744 \\
\hline
\text{Sum (1)} &= 141.3720 \\
\text{Square root of product (2)} &= \sqrt{113.0976 \times 28.2744} \\
&= 56.5488 \\
\text{Volume} &= 6 \times (141.3720 + 56.5488) \\
&= 1{,}187.5248 \text{ cubic inches.}
\end{aligned}$$

EXAMPLES FOR PRACTICE.

1. Find the volume of the frustum of a cone having a slant height of 6 feet, an upper base 6 feet in diameter and a lower base 8 feet in diameter.

Ans. 228.973 cubic feet.

2. Find the lateral area of the frustum of a rectangular pyramid whose altitude is 8 inches and whose bases are 8 × 8 inches and 14 × 14 inches respectively.

Ans. 375.980 square inches.

3. What is the slant height of a frustum of a cone if the convex area is 502.656 square inches, the upper base 7 inches in diameter and the lower base 13 inches in diameter?

Ans. 16 inches.

SPHERE.

294. A sphere is a solid bounded by a curved surface, every point of which is equally distant from a point within called the center.

295. The *diameter* is a straight line drawn through the center and having its extremities in the curved surface. The *radius* is the straight line from the center to the surface; it is equal to one-half the diameter.

296. A sphere is *tangent* to a plane when the plane touches the sphere in only one point. A plane perpendicular to the extremity of a radius is tangent to the sphere.

297. **Surface.** To find the surface of a sphere, *multiply the square of the diameter by* 3.1416.

Example. Find the surface of a sphere which is 9 inches in diameter.

$9^2 = 81. \quad 81 \times 3.1416 = 254.4696$ square inches.

298. To find the diameter when the surface is given, *divide the surface by* 3.1416 and extract the square root of the quotient.

Example. What is the diameter of a sphere whose area is 113.0976 square feet?

$$\frac{113.0976}{3.1416} = 36. \qquad \sqrt{36} = 6 \text{ feet.}$$

299. Volume. To find the volume of a sphere, *multiply the cube of the diameter by* .5236.

Example. A sphere is 10 inches in diameter; what is the volume?

$10^3 = 1000. \quad 1000 \times .5236 = 523.6$ cubic inches.

Another example. Find the volume of a sphere having a diameter of 7 feet.

$7^3 = 343. \quad 343 \times .5236 = 179.5948$ cubic feet.

300. To find the diameter when the volume is known, *divide the volume by .5236 and find the cube root of the quotient.*

Example. A certain sphere contains 8181.25 cubic feet; what is the diameter?

$$\frac{8181.25}{.5236} = 15625. \quad \sqrt[3]{15625} = 25 \text{ feet.}$$

EXAMPLES FOR PRACTICE.

1. A sphere has an area of 153.9384 square inches. What is the volume?

Ans. 179.5948 cubic inches.

2. How much will a sphere of cast iron weigh if it is 3 inches in diameter, and cast iron weighs .26 pounds per cubic inch?

Ans. 3.6756 pounds.

3. How much will it cost to paint a sphere whose radius is 4¼ feet if it costs 25 cents to paint one square yard?

Ans. $7.07.

4. How much less surface has a sphere which is 12 inches in diameter than a sphere which has a volume of 1.7671 cubic feet?

Ans. 3.927 square feet.

5. What is the volume of a sphere if the area is 28.2744 square inches?

Ans. 14.1372 cubic inches.

A PRACTICAL PROBLEM.

To Find the Steam Space in a Horizontal Boiler.

Suppose the boiler represented in Fig. 11 has the following dimensions: length 12 feet, diameter 50 inches and the water level A B C D 16 inches below the top of the boiler.

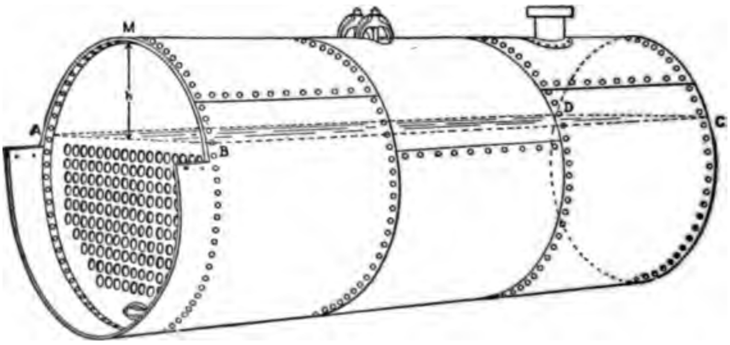


Fig. 11.

The steam space is equal to the volume above the plane A B C D which evidently is equal to the area A M B multiplied by the length of the boiler. To find the area A M B we may use the formula given in Art. 218 since A M B is a segment.

$$A = \frac{4\,h^2}{3}\sqrt{\frac{D}{h} - .608}$$

In the above, $h = 16$ inches, and $D = 50$ inches. Substituting these values we have,

$$\begin{aligned} A &= \frac{4 \times 16^2}{3}\sqrt{\frac{50}{16} - .608} \\ &= 341.33\sqrt{3.125 - .608} \\ &= 542.7 \text{ square inches.} \end{aligned}$$

To reduce this area to square feet, divide by 144.

$$\frac{542.7}{144} = 3.768.$$

The volume equals this area multiplied by the length

$$3.768 \times 12 = 45.216 \text{ cubic feet.}$$

4th. Examination Paper

4th. Examination Paper. "Excellent

EXAMINATION PAPER

Emmie 1-5—
2-5 – 1 sign wrong, 1 inverted.
3-5
4-5 – 1 diff. of decimal
5-5 – 1 ″ ″ , 1 error—
6-5 – 1 error, 1 erroneous
7-5—
8-14

2. $x = \frac{(p \times l) - a\,c}{2}$, when $p = 24$, $l = 30$, $a = 6$ and $c = 10$. $x = 330$.

3. $a\,x = a\,b - 2\,a\,x + 28$, when $a = 2$, and $b = 16$.

4. If $m = 11$, $a = 900$ and $n = 5$, what is the value of c in the following equation? $c = 5$

$$\sqrt{a} = c\,m - n^2.$$

5. If $a = 3$, $b = 6$ and $c = 5$, find the value of d, $= 8$

$$\frac{a^2 + 3\,c}{2} = b\,d - (2\,a^2 + a\,b)$$

EXERCISE IV.

1. If I = 14 and R = 5 what is the value of W from the equation

$$W = I^2 R.$$

2. If S = 9000, E = .85, p = 115 and D = 60 find t from the formula

$$t = \frac{D\,p}{2\,S\,E}. \quad = \frac{23}{57}$$

3. Find the value of E from the formula — 95.52

$$E = \frac{W\,v^2}{2\,g} \text{ if } v = 16,\ W = 24 \text{ and } g = 32.16.$$

4. If S = 887.92 feet and g = 32.16 find the value of t. =

$$S = \tfrac{1}{2}\,g\,t^2.$$

5. Find the value of P from the equation

$$P = \frac{806{,}000\,t^2}{l\,d}. \quad = 161.4583 \quad [161$$

in which $t = .25$, $l = 48$ and $d = 6\frac{1}{2}$.

EXERCISE V.

1. How many square feet in a square 27 rods on each side?

2. The area of a rectangle is 37.8 square feet, and the base is 7 feet. What is the altitude? 5.4

3. The diagonal of a trapezium is 21 feet long, the perpendiculars to this diagonal from the vertices on either side are 9 feet and 8 feet. What is the area?

Ans. 178.5 square feet.

4. If the area of a triangle is 500 square yards and the altitude is 180 feet, what is the base? See ba

5. How many square feet are there in a building lot having two parallel sides 140 feet and 116 feet in length and 83 feet apart?

140
116
256

$\frac{256}{2}$ 83 = 10,624

EXERCISE VI.

1. Find the altitude and area of an isosceles triangle whose base is 20 feet long and the equal sides each 26 feet long. 240

2. The base of a right triangle is 18 feet and the other leg 24 feet. What is the hypothenuse? 30

3. What is the length of the longest rod that, without bending, can be put into a box 1 yard long, 1 foot wide and 9 inches high, measured on the inside? See back

Ans. 3 feet 3 inches.

4. The circumference of a circle is 574.9128 feet. What is the area? See back

5. The area of a circle is 283.5294 square feet. What is the radius? 9.5

EXERCISE VII.

1. Find the entire surface of a rectangular solid whose dimensions are 2, 3 and 4 feet. 52

2. Find the volume of a prism 5 feet high, having for its base a triangle each side of which is 10 feet long. 100 × .433 × 5 = 216.5

3. Find the entire surface and the volume of a cylinder 7 inches in diameter and $1\frac{1}{2}$ feet high. 472.8108 & 692.7228

4. The volume of a sphere is 140,501 cubic inches. What is its diameter? 64.5

5. The radius of a sphere is 28.5 feet. What is the surface? 10207.0584

EXERCISE VIII.

Miscellaneous Problems.

1. At 3 cents per square foot how much would it cost to paint the outside of a cylindrical tank 14 feet high and 16 feet in diameter? Include the cover.

703.7
201.0
904.7

2. How many cubic feet are there in a piece of timber 15 feet long, 16 inches square at one end and 12 inches square at the other?

$27.14

Ans. $20\frac{5}{9}$ cu. ft.

144
256
192
592 × $\frac{15}{3}$ 12 ÷ 1728 = 20.55 or 20 $\frac{960}{17}$

1728 = π = 3.1416

& 3.1416 × 485 = 1523.676 lbs

3. Find the weight of a hollow steel shaft 12 feet long having an outside diameter of 8 inches and an inside diameter of 4 inches. Assume the steel to weigh 485 pounds per cubic foot. (a)

4. Iron being 7.2 times as heavy as water and water weighing 62½ pounds per cubic foot, what is the weight of an iron ball 6 inches in diameter? 29.4524 lbs

5. Find the weight of water contained in a vertical boiler 4 feet in diameter and 10 feet high, the water level being 8 feet from the bottom. Assume 124 tubes each 2 inches in diameter.
Ans. 4930.5 pounds (approx.)

(b) 6. Find the lateral surface of a regular four-sided pyramid 12 feet high, the length of one side of the base being 10 feet. 260

7. The wheel of a bicycle is 28 inches in diameter. How many times will the wheel turn in going a mile? 720.28083

(c) 8. A house that is 50 feet long and 40 feet wide has a pyramidal roof, whose altitude is 15 feet. Find the length of a rafter reaching from a corner of the building to the vertex of the roof.
Ans. 35.35+ ft.

(d) 9. The diameter of a boiler is 6 feet, the length 18 feet. If the maximum depth of the water is 4 feet, what is the volume of the steam space?

10. What is the volume of a frustum of a cone, the altitude of which is 9 feet, and the diameters of the bases 11 feet and 8 feet 7 inches?
Ans. 681.15+ cu. ft.

11. A gallon of water contains 231 cubic inches. How many gallons are there in a cylindrical cistern 11 feet deep and 7 feet in diameter?
Ans. 3,166.7+ gals.

12. What is the amount of surface exposed to the water in the above problem? [illegible]

.7854
2.598

13. What is the difference in area between a circle 8 feet in diameter and a regular hexagon inscribed in it?
Ans. 8.69+ sq. ft.

14. Suppose instead of a circle and regular hexagon we had a regular hexagonal prism inside a cylinder, what would be the difference in volume, the altitude of each being 4 feet 7½ inches?

40.226 cub

(b) $\frac{10}{2} = 5$ $5^2 + 12^2 = 169$ $\sqrt{169} = 13$

$(10 \times 4) \times (13 \div 2) = 40 \times 6.5 = 260$.

(c) $(40 \div 2)^2 = 400$

$15^2 = 225$

sq. lat. h. = 625

〃 ½ long side = 625

sq. lat. edge = 1250 (35.35 ft

(d) Steam space = segment

$\therefore\ A = \frac{4 \times 4}{3}\sqrt{\frac{6}{2} - 0.608}$

$\sqrt{\frac{6}{2} - 0.608} = \sqrt{2.392} = 1.54660$

$3\,)\,1.54660$

$.51553$

16

8.2480

18

148.4640 cub. ft.